Q. Inf. Science 3 (8.S372 / 18.S996) — Fall 2020

Assignment 3

Due: **Friday**, *Sep 25, 2020 at* ***5pm***

Turning in your solutions: Upload a single pdf file (typed or neatly handwritten) to canvas.

1. **Relation between quantum and classical distance measures**
Let $\mathcal{M} = (M_1, \ldots, M_m)$ denote a quantum measurement, meaning that $M_1, \ldots, M_m$ are psd operators that add up to I. Let $\mathcal{M}(\rho)$ denote the resulting probability distribution from measuring a state ρ, i.e.

$$\mathcal{M}(\rho) = (\mathrm{tr}[M_1\rho], \ldots, \mathrm{tr}[M_m\rho]). \tag{1}$$

(a) Prove that measurement can only decrease the trace distance and increase the fidelity, i.e.

$$\frac{1}{2}\|\rho - \sigma\|_1 \geq \frac{1}{2}\|\mathcal{M}(\rho) - \mathcal{M}(\sigma)\|_1 \tag{2}$$

$$F(\rho, \sigma) \leq F(\mathcal{M}(\rho), \mathcal{M}(\sigma)) \tag{3}$$

(b) Show that (2) and (3) can be equalities for the right choice of $\mathcal{M}$ (not the same for each). In other words there exists an $\mathcal{M}$ such that $\frac{1}{2}\|\rho-\sigma\|_1 = \frac{1}{2}\|\mathcal{M}(\rho)-\mathcal{M}(\sigma)\|_1$ and there exists an $\mathcal{M}$ such that $F(\rho, \sigma) = F(\mathcal{M}(\rho), \mathcal{M}(\sigma))$. The second claim, involving fidelity, is harder, so you need to prove this only when ρ is full rank. Hint: try measuring in the eigenbasis of $R := \rho^{-1/2}(\sqrt{\rho}\sigma\sqrt{\rho})^{1/2}\rho^{-1/2}$, and observe that $\sigma = R\rho R$.

2. **Gentle measurement.** Suppose we perform a two-outcome measurement $\{M, I-M\}$ with $0 \leq M \leq I$. This does not uniquely define the post-measurement states, but we will assume that when the first outcome occurs, ρ is mapped to

$$\sigma := \frac{\sqrt{M}\rho\sqrt{M}}{\mathrm{tr}[M\rho]}. \tag{4}$$

(This happens with probability $\mathrm{tr}[M\rho]$.) Quantum measurements can sometimes cause significant disturbance, so it is possible that σ is far from ρ, but this turns out not to happen when $\mathrm{tr}[M\rho]$ is close to 1. Prove that

$$F(\rho, \sigma) \geq \sqrt{\mathrm{tr}\, M\rho}. \tag{5}$$

Hint: Can you show that $\sqrt{M} \geq M$?

3. **Channel fidelity** Consider a compression scheme for ρ, consisting of an encoding $\mathcal{E}$ followed by a decoding $\mathcal{D}$ Let $\mathcal{N} := \mathcal{E} \circ \mathcal{D}$. (In class we considered the case where $\rho = \sigma^{\otimes n}$. For simplicity, in this problem we will just talk about compressing ρ without making use of any tensor power structure.)

   This problem will discuss different accuracy metrics for the compression scheme. We saw that it is not enough for $F(\rho, \mathcal{N}(\rho))$ to be high. Three other possibilities are

   $$\text{entanglement fidelity} \qquad F_e := F(\phi^\rho, (\mathcal{N} \otimes \text{id})\phi^\rho), \tag{6a}$$

   where ϕ^ρ_{AB} is a pure state satisfying $\operatorname{tr}_B \phi^\rho_{AB} = \rho_A$ (it turns out that F_e should not depend on which purification is used;

   $$\text{average fidelity} \qquad \bar{F} := \min \sum_i p_i F(\varphi_i, \mathcal{N}(\varphi_i)), \tag{6b}$$

   where the min is taken over all decompositions of ρ into pure states (not necessarily orthogonal) $\rho = \sum_i p_i |\varphi_i\rangle\langle\varphi_i|$; and

   $$\text{eigenbasis fidelity} \qquad F_\lambda := \sum_i \lambda_i F(\psi_i, \mathcal{N}(\psi_i)), \tag{6c}$$

   for the eigendecomposition of ρ into $\sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$. Prove that

   $$F_e \le \bar{F} \le F_\lambda. \tag{7}$$

   Find an example where F_λ is much higher than $\bar{F}$. It turns out that if $\bar{F} = 1 - \epsilon$ then $F_e \geq 1 - \frac{3}{2}\epsilon$ but this is not easy to prove, and is not a required part of the pset. Some discussion of this point will be in the pset solutions, and feel free to work on the question yourself if you like.

4. **Entropy inequalities.**

   (a) If $|\psi\rangle_{AB}$ is a pure state show that $S(A) = S(B)$.

   (b) *Triangle inequality.* If $|\psi\rangle_{ABC}$ is pure, then $S(A) \leq S(B) + S(C)$.

   (c) *Araki-Lieb inequality.* For a general state ρ_{AB},

   $$|S(A) - S(B)| \leq S(AB). \tag{8}$$

   (d) $|S(A|B)| \leq S(A)$.

   (e) $I(A : B|C) \leq 2 \log \min(\dim A, \dim B)$

   (f) *Optional.* Finish the proof that $D(\rho\|\sigma) \geq 0$. Hint: The concavity of log means that $\sum_j p_j \log(x_j) \leq \sum_j \log(p_j x_j)$.