Q. Inf. Science 3 (8.S372 / 18.S996) — Fall 2020

Assignment 9

Due: **Friday**, *Dec 4, 2020 at* ***5pm*** on canvas.

1. **Monogamy of entanglement**

   (a) The principle of *monogamy of entanglement* is that entanglement cannot be shared without limit, unlike classical correlations. However, the larger the local dimension, the more systems can be be simultaneously entangled. We will start with an example of this phenomenon. Let

   $$|\psi\rangle_{A_1,\ldots,A_n} = \sum_{\pi \in S_n} \mathrm{sgn}(\pi) |\pi_1\rangle \otimes |\pi_2\rangle \otimes \cdots |\pi_n\rangle \in (\mathbb{C}^n)^{\otimes n}. \tag{1}$$

   Here S_n is the symmetric group, meaning the set of $n!$ permutations of n objects. The sign of a permutation $\mathrm{sgn}(\pi) = (-1)^m$ where m is the number of transpositions (swaps of two elements) in any decomposition of π. We will show that $\psi_{A_1A_2}$ is far from $\mathrm{Sep}(n,n)$. To show this, let $M = (I - \mathrm{swap})/2$. Show that $\mathrm{tr}[M\psi_{A_1A_2}] = 1$ and $\mathrm{tr}[M\sigma] \leq 1/2$ for any $\sigma \in \mathrm{Sep}(n,n)$.

   (b) Despite the above example, nontrivial statements about monogamy can be made when the number of systems is only logarithmic in the local dimension. This will follow from some information-theory tools that we now develop. Let $I(A : B|X)_\rho = \epsilon$ and suppose that X is classical while A, B are quantum. Show that there exists a separable state σ_{AB} such that $\frac{1}{2}\|\rho_{AB} - \sigma_{AB}\|_1 \leq \sqrt{\epsilon \ln(2)}$. (Hint: you should review pset 6, problem 1(b)

   (c) Consider the state $\rho^{AB_1\ldots B_k}$, where A has dimension d_A and each B_i has dimension d_B. Let $\{M, I-M\}$ be a 1-LOCC measurement on two systems A and B, meaning that it can be written as a measurement on system B followed by a two-outcome measurement on system A, i.e.

   $$M = \sum_{y=1}^{m} Q_y \otimes R_y, \tag{2}$$

   where each $R_y \geq 0$, $\sum_{y=1}^m R_y = I$ and $0 \leq Q_y \leq I$. It turns out that most of the ρ^{AB_i} are close to Sep when measured with M of this form. To see this, consider the state $\sigma^{AY_1\ldots Y_k}$ where we measure each system B_i for $i = 1, \ldots, k$ using the measurement $\{R_1, \ldots, R_m\}$ and we record the answer in a classical system Y_i. Let $Y_{<i} := Y_1Y_2, \ldots, Y_{i-1}$. Show that

   $$\sum_{i=1}^{k} I(A : Y_i|Y_{<i})_\sigma \leq \log(d_A). \tag{3}$$

(d) Given ρ and M as above, define $h_{\text{Sep}}(M) = \max\{\operatorname{tr} M\sigma : \sigma \in \text{Sep}(d_A, d_B)\}$. Show that

$$\mathop{\mathbb{E}}_{i\in[k]} \operatorname{tr}\left[M\rho^{AB_i}\right] \leq h_{\text{Sep}}(M) + \sqrt{\frac{2\ln(d_A)}{k}}. \tag{4}$$

This shows a nontrivial monogamy relation when the number of systems is only logarithmic in the local dimension. On the other hand, it applies only to a restricted family of measurements. Hint: you may want to relate $I(A : Y_i|Y_{<i})$ to the states of ρ resulting from measuring some subsystems and conditioning on the outcomes, while leaving other systems unmeasured or traced out.

2. **The Toric Code** An exciting avenue for the realization of fault-tolerant quantum computation is through the encoding of bits in many-body 'topologically ordered' systems. This was first proposed by Kitaev in a seminal paper; here we walk through a few of the basic ideas (figures are also from that paper). This problem is optional for those who signed up for extra scribing.

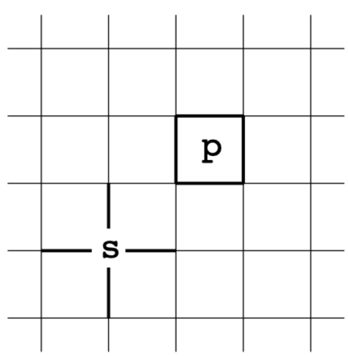
We work with a $k \times k$ square lattice on a torus, and place a qubit on each *link* of the lattice. Consider the following operators:

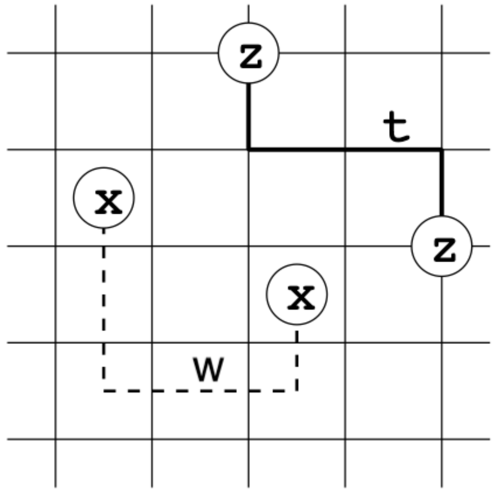
$$A_s = \prod_{j\in\text{star}(s)} X_j \qquad\qquad B_p = \prod_{j\in\text{boundary}(p)} Z_j \tag{5}$$

which are defined in the star of a lattice site s or the boundary of a lattice plaquette p. We will be interested in states $|\psi\rangle$ with $A_s|\psi\rangle = B_P|\psi\rangle = |\psi\rangle$, or, equivalently, the ground state dynamics of the Hamiltonian:

$$H = -\sum_s A_s - \sum_p B_p \tag{6}$$

as well as states that violate this condition minimally. *Hint: Drawing pictures will make most of these problems far simpler, and proof-by-carefully-explained-pictures is encouraged.*





(a) Show that $[A_s, B_p] = 0$, and determine the allowed eigenvalues of each.

(b) There is a convenient, pictographic way to understand the ground states of this system. Working in the $|\pm x\rangle$ basis, consider any qubit j to be 'unoccupied' if $X_j = +1$, and 'occupied' if $X_j = -1$. Show that any state $|\psi\rangle$ with $A_s |\psi\rangle = |\psi\rangle$ is a superposition of states where the 'occupied' sites form closed loops. Pictures strongly encouraged.

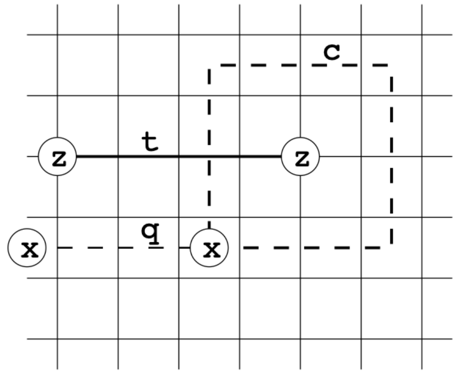
(c) What happens when the operators B_p act on one of the closed loops states? Argue that any state $|\psi\rangle$ with $A_s |\psi\rangle = B_p |\psi\rangle = |\psi\rangle$ must be an equal weight superposition of all contractible closed loop states. We will call these states ground states.

(d) However, there remain two non-contractible closed loops around the torus, and the behavior of the ground states around these loops is undetermined. To show this, let t be a string along links and sites of the lattice, and w a string of plaquettes and sites (see figure). We define the string operators:

$$S^z(t) = \prod_{j \in t} Z_j \qquad\qquad S^x(w) = \prod_{w \in t} X_j \tag{7}$$

Let t, w be closed loops, including if they go around a non-contractible loop of the torus, and let $|\psi\rangle, |\theta\rangle$ be equal-weight superposition of closed loop states as above. Show that $\langle\psi| [S^z(t), H] |\theta\rangle = \langle\psi| [S^x(w), H] |\theta\rangle = 0$. Show that if t or w is a contractible loop then $S^z(t), S^x(w)$ act trivially on the ground states. Argue that they act non-trivially on the ground state subspace if t, w are non-contractible loops.

Continuing to work in the $|\pm x\rangle$ basis, where the A_s operators are most easily diagonalized, the four ground states of this system may be labeled by $\nu_1 := S^x(\gamma_1) =$



$\pm 1, \nu_2 := S^x(\gamma_2) = \pm 1$, where γ_1, γ_2 are loops around the two nontrivial cycles of the torus. Each of the ground states is an equal weight sum over all contractible closed-loop configurations, but includes non-contractible loops as determined by ν_1, ν_2. Argue that no local measurement, i.e. any operator involving qubits at most a distance $\ell << k$ from each other, can distinguish among these four ground states.

(e) Now we are interested in excited states, i.e. states $|\phi\rangle$ which do not have $A_s |\phi\rangle = B_p |\phi\rangle = |\phi\rangle$. Naïvely, our first guess would be to find a state which violates a single A_s or B_p constraint. However, since $\prod_s A_s = \prod_p B_p = 1$, this is not possible. Instead, we must violate two A_s or B_p constraints.

Let t, w be strings with two endpoints. Show that $S^z(t) |\nu_1, \nu_2\rangle$ violates the A_s constraints at the endpoints (and nowhere else) and $S^x(w) |\nu_1, \nu_2\rangle$ violates the B_p constraints at the endpoints (and nowhere else).

Argue that that if t, t' are two loops with the same endpoints, that may be smoothly deformed into each other (i.e. they don't go off around non-trivial cycles of the torus in different ways), then $S^z(t) |\nu_1, \nu_2\rangle = S^z(t') |\nu_1, \nu_2\rangle$, so that the string is not physical.

The endpoints of the strings are excitations, and should be regarded as physical particles in this theory. They may move, hop around, and annihilate. We refer to the ends of $S^z(t)$ as e particles and the ends of $S^x(w)$ as m particles.

(f) Suppose that there is an e particle at a site j, and let t be a string from j to j'. Argue that $S^z(t)$ annihilates the e particle at j and creates a e particle at j'. (in other words it hops the e particle from j to j'.)

The bizarre and useful aspects of topologically ordered states is that the particles have non-trivial statistics. Ordinarily, dragging two different species of particles around each other does not yield any exchange statistics. Suppose that we create two e particles and two m particles, as shown in the figure, connected by strings t, q. The initial state is then:

$$|\Psi\rangle = S^z(t) S^{(}x)(q) |\nu_1, \nu_2\rangle \tag{8}$$

Now, we hop the m particle around the e particle by applying $S^x(c)$, where c is a closed loop. Show that:

$$S^x(c)\,|\Psi\rangle = -\,|\Psi\rangle \tag{9}$$

The phase here is not due to any dynamics or energy, but rather due to the topological nature of the wavefunction. It arises because these excitations are not bosons, nor fermions, but anyons.

(g) Finally, we arrive at the application of a quantum gate. Suppose we allow there an excitation in some region of our toric code (one may add extra excitations outside this region to account for the fact that $\prod_s A_s = \prod_p B_p = 1$). There are four possibilities for these excitations:

   i. Trivial. There's no particle there, and we label the lack of an excitation by 1
   ii. There is an e particle there, which we denote by e.
   iii. There is an m particle there, which we denote by m.
   iv. There are both an e and an m particle there, which we denote by em.

One needs to 'course grain' the lattice model slightly to allow both an e and an m to be present - think of this as the region of a sample around the end of a probe. Note that as two e or two m particles may always annihilate, we exclude cases with two or more e or m particles.

Now suppose we have two such regions, say near the end of two probe tips. Hence we have a set of states $|1,1\rangle\,, |e,1\rangle\,, ..., |em, em\rangle$. Let $\mathcal{B}$ (braiding) be the operation of dragging one of these excitation regions around the other, as in the previous problem. What are the matrix elements of $\mathcal{B}$ acting on this 16 dimensional space? (Most of these are zero; just give the values of the nonzero elements).

This problem walked us through the toric code, in particular the non-trivial braiding operation. One extraordinary aspect of this model is that the qubits are completely protected. Just as the ground states are locally indistinguishable, no isolated e or m particle may be annihilated by local operators. And one may perform braiding among many excitation regions to execute a wide array of gates.

However, all the braiding operators in the toric code are diagonal, and so all the gates we can perform (even with many excitation regions) commute - surely this is not sufficient for universal computing. However, there exist other 'topological' orders which enjoy the same protections, but have an additional property: when particles are dragged around each other, the particle type changes (this would be like an e changing to an m after braiding). These nonabelian topological orders are sufficient for universal computation, and an active area of research is proposing new physical materials and implementations to perform this. Recent interest has been focused around the edges of nanowires and the cores of vortices in topological superconductors.