8.S372/18.S996 Quantum Information Science III	**Fall 2020**

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1.0.1 Entanglement and Density Matrices

In quantum mechanics (QM), a *pure state* $|\psi\rangle$ is a vector in $\mathbb{C}^2, \mathbb{C}^d, \mathbb{C}^2 \otimes \mathbb{C}^2$, etc, that we use to describe a system whose state is known. On the other hand, a *mixed state* is when a system is a statistical mixture of pure states, and must be described by a *density matrix*:

$$\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i| \in H(\mathbb{C}^d) \tag{1.1}$$

Here the system is in state $|\psi_i\rangle$ with probability p_i. Note that any pure state $|\psi\rangle$ has a density matrix representation as $|\psi\rangle\langle\psi|$. We shall use the notation $\psi \equiv |\psi\rangle\langle\psi|$.

As a general rule, density matrices are Hermitian matrices such that $\text{Tr}\ \rho = 1$ and all eigenvalues are nonnegative (often written as $\rho >= 0$). One should think of this as the quantum analog of the probability simplex, and indeed there are several notions of a probability distribution encoded in a density matrix ρ. If we measure a system in the natural bases $(|1\rangle \dots |d\rangle)$, then ρ_{ii} is the probability to find the system in the state $|i\rangle$. So we may think of the diagonal entries of ρ as a probability distribution. This holds true even if we change basis, and so the eigenvalues of ρ are again a probability distribution.

Mixed states can be obtained from entangled states by discarding information about a subsystem. Let our system partition into A and B subsystems. Then $\psi_A \equiv \text{Tr}_B \psi$. Specifically, suppose that we can write a pure state $|\psi\rangle$ as:

$$|\psi\rangle = \sum_{ij} c_{ij} |i\rangle \otimes |j\rangle \tag{1.2}$$

(we will sometimes omit tensor product symbols below). Then we can write the density matrix as:

$$\psi = \sum_{ijkl} c_{ij} c^*_{kl} |i\rangle \otimes |j\rangle \langle k| \otimes \langle l| = \sum_{ijkl} c_{ij} c^*_{kl} |i\rangle\langle k| \otimes |j\rangle\langle l| \tag{1.3}$$

Now, note that we may consider $\text{Tr} : L(\mathbb{C}^d) \to \mathbb{C}$, and $I : L(\mathbb{C}^d) \to \mathbb{C}^d$. So that $\text{Tr}_B = \text{Tr} \otimes I$. Accordingly, taking the trace over the B subsystem above replaces

$\text{Tr}\,(|j\rangle\langle l|) = \delta jl$, and so:

$$\psi_A = \text{Tr}\,_B\psi = \sum_{ijk} c_{ij}c^*_{kj}\,|i\rangle\,|j\rangle \tag{1.4}$$

If we consider c_{ij} to be the entries of a (not necessarily square) matrix C, then we can write this as:

$$\psi_A = CC^\dagger \tag{1.5}$$

Examples:

1. Suppose that C is rank 1, or equivalently that $c_{ij} = \alpha_i\beta^*_j$. Then ψ is an unentangled product state, and $\psi_A = \alpha_i\alpha^*_j = |\alpha\rangle\langle\beta|$. Later we will see that ψ is a product state $\leftrightarrow$ ψ_A is pure state $\leftrightarrow$ ψ_B is a pure state.

2. Suppose that C has the form $C = \frac{1}{d}U$, where U is a unitary matrix, and d is the dimension of the matrix (necessary so that $\text{Tr}\,\psi = 1$). We can write $U = \sum_i |u_i\rangle\langle u_i|$, where u_i are the orthonormal eigenvectors of U. Then we can write the quantum state as:

$$|\psi\rangle = \frac{1}{\sqrt{d}}\sum_i |u_i\rangle\,|i\rangle \tag{1.6}$$

   Then one can check that

$$\psi_A = \frac{1}{d}\sum_i |u_i\rangle\langle u_i| \tag{1.7}$$

These two examples display the range of information that can be lost when we throw out a subsystem. In the first example, the A subsystem remains in a pure quantum state, despite the loss of B. On the other hand, discarding the B system destroys all correlation in A in the second example, leaving A in a "fully mixed state."

These phenomena are related to the *singular value decomposition* (SVD) of the matrix C. Let $C = UDV^\dagger$, with U, V unitary and D diagonal. Note that $\psi_A = CC^\dagger = UD^2U^dagger$. this implies that the eigenvalues of ψ_A are the squares of the singular values of C ($\text{eig}(A) = \text{svd}(C)^2$). **Exercise:** Show that $\text{eig}(\psi_A) = \text{eig}(\psi_B)$. This also implies that ψ_A does not depend on V. **Exercise:** Show that ψ_A is independent of unitary transformations on the B subsystem, and vice-versa.

1.0.2 Purifications (to be continued)

. The basic idea of a purification is to construct a pure state from a density matrix. Given some ρ on a system A, can we add some subsystem B and create a state $|\psi\rangle$ on systems A and B so that $\psi_A = \text{Tr}\,_B\psi = \rho$? In the next class, we will show that this is always possible, but not unique. This will lead to interesting results regarding bit commitment.