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14.1 Quantum Capacity formula

For a given quantum channel $N_{A\to B}$, consider the definition of an environment system under the Stinespring representation $N(\rho) = \mathrm{Tr}_E[V\rho V^\dagger]$. We define the **coherent information** as

$$I_C(\rho, N) = S(N(\rho)) - S(N^c(\rho)) = S(B) - S(E) \tag{14.1}$$

Where the superscript c denotes tracing out the complement subspace. For example, if in $N(\rho)$ we trace out subspace E, then in $N^c(\rho)$ we trace out subspace B. Under this definition, we can now define the **quantum capacity** according to the LSD theorem (Lloyd, Shor, Devetak):

$$Q(N) = \lim_{n\to\infty} \frac{1}{n} \max_\rho I_c(\rho, N^{\otimes n}) \tag{14.2}$$

$$= \max_\rho I_c(\rho, N) \qquad \text{if } N \text{ is degradable} \tag{14.3}$$

Let us consider the definition of I_c through the purification of the initial state. In particular, let $\phi_{AA'}$ be a pure state, and consider the subsystems B, E resulting of feeding A' through the channel N. Under the tri-partite state τ_{ABE},

$$I_c = S(B)_\tau - S(E)_\tau = S(B) - S(AB) = -S(A|B) = \frac{I(A:B) - I(A:E)}{2} \tag{14.4}$$

Likewise we can consider the entanglement of distillation $E_D(\rho)$

$$E_D(\rho_{AB}) = \lim_{n\to\infty} \frac{1}{n} \max_{\Lambda: A_1\cdots A_n \to AE'} [S(B_1\cdots B_n) - S(AB_1\cdots B_n)] \tag{14.5}$$

We can define metric of capacity using an arbitrary penalty on the conditional entropy between A and B called the **Hare-brained capacity**:

$$C_{\mathrm{HB}}(N) = \max H(B) - 10H(B|A) \tag{14.6}$$

While this penality is arbitrary, it still satisfies

$$\lim_{n\to\infty} \frac{1}{n} \max_\rho C_{\mathrm{HB}}(N^{\otimes n}, \rho) = C(N) \tag{14.7}$$

14.2 Understanding the Capacity Formula

1. N is antidegradable. If $E = E_B E_E$, $I(A:E) = I(A:E_B) + I(A:E_E|E_B) \geq I(A:E_B) = I(A:B)$

   $I_C \leq 0$. Though this statement does not hold if allowing feedback.

2. Consider a random Pauli channel, which applies a random Pauli matrix with some respective probability. $N(\rho) = (1 - p_x - p_y - p_z)\rho + p_x X\rho X + p_y Y\rho Y + p_z Z\rho Z$

   Applying this channel,

   $$(I \otimes N)\Phi = p_I \Psi_0 + p_x \Psi_1 + p_y \Psi_2 + p_z \Psi_3 \tag{14.1}$$

   where $|\Psi_i\rangle = (I \otimes \sigma_i)|\Psi\rangle$

   A purification of this density matrix is the wavefunction

   $$\sqrt{p_I}\,|\Psi_0\rangle_{AB}|0\rangle_E + \sqrt{p_X}\,|\Psi_1\rangle_{AB}|1\rangle_E + \sqrt{p_Y}\,|\Psi_2\rangle_{AB}|2\rangle_E + \sqrt{p_Z}\,|\Psi_3\rangle_{AB}|3\rangle_E \tag{14.2}$$

   which has $S(B) = 1$ and $S(E) = H(\vec{p})$.

   A special case of this is the **depolarizing channel** D_p, with $S(E) = H_2(p) + p\log 3$.

   ”Hashing bound” cf. hashing method to check if $x \overset{?}{=} y$. We choose some random function $f \to \{0,1\}^k$ and check if $f(x) \overset{?}{=} f(y)$. If $x = y$, we necessarily have $f(x) = f(y)$, and if $x \neq y$, the probability that $f(x) = f(y)$ can be be shown to be small $\Pr[f(x) = f(y)] \sim 2^{-k}$

3. Sometimes preprocessing helps

   $$\rho = \Phi_{A,B} \otimes \left(\frac{I}{2}\right) A_2 \tag{14.3}$$

   $I_C = 0$, preprocessing results in $\to 1$ or $I_C(I/2, D_p) = 1 - H_2(p) - p\log 3$.

4. Sometimes entangled inputs help. `quant-ph/9706061` for $p \approx 0.19$, $I_C(I/2, D_p) < \frac{1}{5} I_C(\frac{|00000\rangle\langle 00000| + |11111\rangle\langle 11111|}{2}, D_p^{\otimes 5})$

5. **Superactivation** $\exists N_1, N_2$ s.t. $Q(N_1) = Q(N_2) = 0$, but $Q(N_1 \otimes N_2) > 0$

   An example of this is with $N_1 = 50\%$ erasure channel and N_2 a PPT channel with private $C \propto \rho > 0$(???). This satisfied $Q(N_1 \otimes N_2) > 0$ according to `Smith-Yard, Science 2008, arxiv:0807.4935`.

14.3 PPT Channels

The **partial transpose** of a density matrix is defined as

$$\rho^\Gamma = (I \otimes T)\rho \tag{14.1}$$

Where for example

$$(|i\rangle\langle j| \otimes |k\rangle\langle l|)^\Gamma = |i\rangle\langle j| \otimes |l\rangle |k\rangle \tag{14.2}$$

Let us start to build some intuition on this operation. Let PPT be the set of density matrices that under partial transpose remain positive semi-definite, i.e.

$$\text{PPT} = \{\rho : \rho^\Gamma \geq 0\} \tag{14.3}$$

Since all psd matrices are symmetric, $\rho^T = \rho$ is psd. Thus, Sep $\subseteq$ PPT. On the pset, you will show that if $\rho \in D_{d^2}$ and $\rho^\Gamma \geq 0$, then Tr $[\rho^\Gamma \Phi_d] \leq 1/d$. Let us now consider the composition of the partial transpose and LOCC operations:

Claim If $\rho \in$ PPT, and $\mathcal{E}$ is a LOCC operation, then $\mathcal{E}(\rho) \in$ PPT. This result extends to SLOCC (stochastic LOCC).

To quickly recap some definitions, local operations and classical communication channels are described by

$$\rho \to (U \otimes V)\rho(U \otimes V)^\dagger \tag{14.4}$$

Measurement channels,

$$\sum_k (E_k \otimes I)\rho(E_k \otimes I)^\dagger \text{ with } \sum E_k^\dagger E_k \leq \mathbb{I} \tag{14.5}$$

and stochastic LOCC

$$\rho \to (E_k \otimes I)\rho(E_k \otimes I)^\dagger \text{ or } (I \otimes E_k)\rho(I \otimes E_k)^\dagger \tag{14.6}$$

In general,

$$\rho \to (A \otimes B)\rho(A \otimes B)^\dagger \tag{14.7}$$

Where A, B are arbitrary or perhaps invertable.

$$((A \otimes B)\rho(A \otimes B)^\dagger)^\Gamma = (A \otimes \bar{B})\rho^\Gamma(A \otimes \bar{B})^\dagger \geq 0 \qquad \text{if } \rho^\Gamma \geq 0 \tag{14.8}$$

Where is this useful? D_p is never antidegradable for $p < 1$, but is PPT for p large enough.

$\rho \in$PPT $\implies E_{D,2}(\rho) = 0$. PPT = Sep only if $d_A = 2, d_B = 3$.

14.3.1 States

As previously argued, there is a straightforward inclusion statement between the set of separable states and PPT

$$\text{Sep} \subset \text{PPT} \subset \text{All} \tag{14.9}$$

Doherty, Parrilo and Spedalieri defined the DPS hierarchy (quant-ph/0308032), based on iteratively extending approximations to the set of entangled states.

$$\text{PPT} = \text{DPS}_1 \supset \text{DPS}_2 \supset \cdots \supset \text{DPS}_\infty = \text{Sep} \tag{14.10}$$

Each DPS_k requires time $d^{\mathcal{O}(k)}$ to search, so there is a tractable test for entanglement that increases exponentially with k.

14.3.2 Operations

$$1 - \text{LOCC} \supset \text{LOCC} \supset \text{Sep} \supset \text{PPT} \supset \text{All} \tag{14.11}$$

PPT operations map PPT states onto PPT states. PPT channels always output PPT states.