8.S372/18.S996 Quantum Information Science III	Fall 2020

Lecture 18: November 3, 2020

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Recall from last time that

$$\mathop{\mathbb{E}}_{|v\rangle \in \mathbb{C}^d} |v\rangle \langle v|^{\otimes n} = \frac{\Pi_{sym}^{d,n}}{\mathrm{tr}\Pi_{sym}^{d,n}} = \frac{\sum_{\pi \in S_n} P_\pi}{d(d+1)\cdots(d+n-1)}.$$

Previously we proved this using direct calculation, but it can be shown in a more illuminating way using representation theory.

18.1 Representation theory

Let G be a group, and V a vector space. Then a map $r : G \to L(V)$ is a representation if

$$r(gh) = r(g)r(h)$$

for all $g, h \in G$.

Examples: $g \in U_d \to g^{\otimes n} \in U(\mathbb{C}^{d^n})$

$\pi \in S_n \to P_\pi \in U(\mathbb{C}^{d^n})$ (acts by permuting the positions).

Fix $\omega \in \mathbb{C}$ where $|\omega| = 1$. Then $z \in \mathbb{Z} \to \omega^z \in U(1)$ is a representation. This also works for any $\omega \in \mathbb{C}$, but we may get a non-unitary representation. Similarly, if $\omega^p = 1$, then $z \in \mathbb{Z}_p \to \omega^z \in U(1)$ is a representation.

Two representations (r_1, V_1) and (r_2, V_2) are considered *equivalent* if there exists $T \in L(V_1, V_2)$ such that

$$Tr_1(g) = r_2(g)T$$

for all $g \in G$. Written as $(r_1, V_1) \cong (r_2, V_2)$.

Fact: If G is finite or compact then any representation is equivalent to a unitary representation.

A representation (r, V) is *reducible* if $(r, V) \cong (r_1 \oplus r_2, V_1 \oplus V_2)$. There is a basis in which for all g, $r(g)$ can be written in block diagonal form $\left(\begin{array}{c|c} r_1(g) & 0 \\ \hline 0 & r_2(g) \end{array}\right)$. If there is no such decomposition, (r, V) is an *irreducible* representation, or irrep.

Examples: Trivial representation $r(g) = 1 \in U(1)$ for all g.

For a finite group G, let $\mathbb{C}[G] = \text{span}\{|g\rangle : g \in \mathbb{C}\} \cong \{f : G \to \mathbb{C}\}$. Then we obtain the left/right regular representations: $L(x)|g\rangle = |xg\rangle$ and $R(x)|g\rangle = |gx^{-1}\rangle$. These representations are reducible, since $\sum_{g\in G}|g\rangle$ is acted on trivially.

In fact, we can decompose $\mathbb{C}[G]$ into irreps. Let $\hat{G}$ be the set of inequivalent irreps (r_λ, V_λ). Then we can write

$$\mathbb{C}[G] \underset{L(g_1)R(g_2)}{\cong} \bigoplus_{\lambda\in\hat{G}} V_\lambda \otimes V_\lambda^* \underset{L}{\cong} \bigoplus_{\lambda\in\hat{G}} V_\lambda \otimes \mathbb{C}^{\dim V_\lambda}$$

where the dual representation (r^*, V^*) is defined as $r^*(g) = r(g^{-1})^T$ and the left representation and right representation act on different spaces. Note that the dimensions of both sides are equal: $|G| = \sum_\lambda d_\lambda^2$ where d_λ is the dimension of V_λ.

We can write $L(V,W) \cong V^* \otimes W$ since linear maps look like $\sum_{v,w} c_{v,w}|w\rangle\langle v|$. If we have two representations (r, V) and (s, W) we obtain a representation $r(g^{-1})^T \otimes s(g) = r^*(g) \otimes s(g)$ acting on matrices as follows:

$$M \in L(V,W) \to s(g)Mr(g)^{-1}$$

since $vec(AMB) = (A \otimes B^T)vec(M)$.

Examples: For unitaries U, $r(U) = U \otimes U^*$ corresponds to $\rho \to U\rho U^\dagger$. A 1D invariant subspace is spanned by the maximally entangled state $|\Phi\rangle = \sum_i |i\rangle \otimes |i\rangle = vec(I)$. The remaining (d^2-1)-dimensional subspace also turns out to be irreducible.

$r(U) = U \otimes U$. This commutes with $F = \text{SWAP}$, so it preserves the V_{sym} and V_{anti} subspaces, which have dimensions $d(d\pm1)/2$. These subrepresentations are irreducible. For $d = 2$, these are known as the triplet and singlet states. In general $(r(U) = U^{\otimes n}, \text{Sym}^n\mathbb{C}^d)$ is an irrep of U_d.

Proof sketch: Suppose $|\psi_1\rangle, |\psi_2\rangle \in \text{Sym}^n\mathbb{C}^d$. There exist $|\phi_1\rangle, |\phi_2\rangle$ such that $\langle\psi_1|^{\otimes n}|\psi_i\rangle \neq 0$. This can be used to show the existence of U such that $\langle\psi_1| r(U) |\psi_2\rangle \neq 0$.

Schur's Lemma: If V_μ, V_ν are irreps of a group G over $\mathbb{C}$, and then the set of G-invariant maps from $V_\mu \to V_\nu$ is

$$L(V_\mu, V_\nu)^G = \begin{cases} 0, & \mu \neq \nu \\ \mathbb{C}I, & \mu = \nu \end{cases}$$

This is the set of maps that preserve the group action, i.e. $r_\nu(g)T = Tr_\mu(g)$ for all g.

Proof: Suppose $T \in L(V_\mu, V_\nu)^G$. Then the subspaces $\ker T$ and $\operatorname{Im} T$ are G-invariant. Since V_μ and V_ν are irreps, either $\ker T = 0$ or $\ker T = V_\mu$, and either $\operatorname{Im} T = 0$ or $\operatorname{Im} T = V\nu$. Therefore, if $\mu \neq \nu$ we must have $T = 0$. Otherwise, if $\mu = \nu$, choose eigenvalue λ of T (which exists since it is over $\mathbb{C}$). Then ker $(T - \lambda I) \neq 0$, so ker $(T - \lambda I) = V\mu$ and $T = \lambda I$.

This does not work for irreps over $\mathbb{R}$. Consider the $SO(2)$ action on $\mathbb{R}^2$, and let $T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. This commutes with the group action, but is not a multiple of I.

Before we move on, we will introduce the Haar measure. This is the uniform measure on compact groups, and is the unique measure satisfying

$$\mu_{\text{Haar}}(S) = \mu_{\text{Haar}}(gS) = \mu_{\text{Haar}}(Sg).$$

If $U \sim$ Haar, then for arbitrary $|v\rangle$, $U|v\rangle$ is uniformly random.

$$\mathop{\mathbb{E}}_{U\sim\text{Haar}} r(U)$$

is the projector onto U_d-invariant vectors.

Calculation:

Let

$$M = \mathop{\mathbb{E}}_{|\psi\rangle\in\mathbb{C}^d,\langle\psi|\psi=1\rangle} |\psi\rangle\langle\psi|^{\otimes n} = \mathop{\mathbb{E}}_{U\sim\text{Haar}} (U|0\rangle\langle 0|U^\dagger)^{\otimes n}) = \mathbb{E}\, r(U)|0\rangle\langle 0|^{\otimes n} r(U)^\dagger.$$

Then

$$r(V)M = \mathop{\mathbb{E}}_{U} r(VU)|0\rangle\langle 0|^{\otimes n} r(U^\dagger) \tag{18.1}$$
$$= \mathop{\mathbb{E}}_{W=VU} r(w)|0\rangle\langle 0|^{\otimes n} r(W \dagger V) \tag{18.2}$$
$$= Mr(V), \tag{18.3}$$

so by Schur's Lemma

$$M = \lambda I_{\text{Sym}^n\mathbb{C}^d} = \frac{\Pi^{d,n}_{\text{Sym}}}{\text{tr}\Pi^{d,n}_{\text{Sym}}}.$$

What about $\mathbb{E}\, U^{\otimes n} M (U^\dagger)^{\otimes n}$ if $M \neq |\psi\rangle\langle\psi|^{\otimes n}$?

Isotypic decomposition: Compact/finite groups satisfy complete reducibility, which means that every representation can be decomposed into a direct sum of irreps.

$$V \cong \bigoplus_{\lambda\in\hat{G}} V_\lambda \otimes \mathbb{C}^{M_\lambda} \tag{18.4}$$
$$\cong \bigoplus_{\lambda} V_\lambda \otimes L(V_\lambda, V)^G \tag{18.5}$$

where $M_\lambda \geq 0$ is the multiplicity of λ.

18.2 Schur-Weyl duality

Duality between action of the unitary group and the symmetric group.

Let $q_n(U) = U^{\otimes n}$ and $p_d(\pi) = \sum_{i_1 \dots i_n \in [d]} |i_1 \dots i_n\rangle \langle i_{\pi(1)} \dots i_{\pi(n)}|$, where both operators act on $(\mathbb{C}^d)^{\otimes n}$. We have $[q_n(U), p_d(\pi)] = 0$.

We can write

$$(\mathbb{C}^d)^{\otimes n} \cong^{U_d \times S_n}_{p_d q_n} \bigoplus_{\lambda \in \mathrm{Par}(n,d)} Q_\lambda \otimes P_\lambda$$

where Q_λ and P_λ are U_d and S_n irreps respectively, labeled by partitions from the set

$$\mathrm{Par}(n,d) = \{\lambda \in \mathbb{Z}^d : \lambda_1 \geq \cdots \geq \lambda_d \geq 0, \sum_i \lambda_i = n\}.$$

Schur-Weyl duality says that the each λ has multiplicity 1.

Follows from the following. Let

$$A = \mathrm{span}\{q_n(U) : U \in U_d\}, B = \mathrm{span}\{p_d(\pi) : \pi \in S_n\}$$

Then $\mathrm{Comm}(A) = \{X : [X,a] = 0 \ \forall a \in A\} = B$ and $\mathrm{Comm}(B) = A = \mathrm{span}\{X^{\otimes n} : X \in M_d\}$ by the Double Commutant theorem.

Examples:

We can denote a partition by a *Young diagram*, where we arrange n boxes in rows corresponding to each part. If $n = 2$, there are two partitions: $\lambda = (2,0)$ $\ydiagram{2}$ and $\lambda = (1,1)$ $\ydiagram{1,1}$ and we have

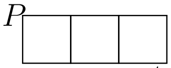
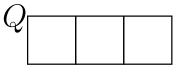
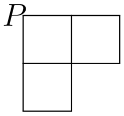
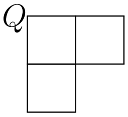
$$P_{\ydiagram{2}} = \text{trivial}$$

$$Q_{\ydiagram{2}} = \text{symmetric}$$

$$P_{\ydiagram{1,1}} = \text{sign}$$

$$Q_{\ydiagram{1,1}} = \text{antisymmetric}$$

If $d = 2$, then for a partition $\lambda = (\lambda_1, \lambda_2)$ we obtain a spin-J representation, where $J = (\lambda_1 - \lambda_2)/2$.

When $n = 3, d = 2$ there are two partitions: P  $=$ trivial, Q  $=$ spin $3/2, \dim 4$ and $\dim P$  $= 2, Q$  $=$ spin $1/2, \dim 2$. The dimensions match up since $2^3 = 1 \cdot 4 + 2 \cdot 2$.

$\#\text{Par} \sim n^d$, $\dim Q_\lambda \leq n^{d^2}$, $\dim P_\lambda \approx \exp(nH(\frac{\lambda}{n}))$.