8.S372/18.S996 Quantum Information Science III **Fall 2020**

Lecture 22: November 17, 2020

Lecturer: Aram Harrow *Scribe: Shankar Balasubramanian*

22.1 Clarifications in previous proof

An important identity that we made use of is

$$\text{Tr}(\text{cycle}) = \sum_{i_1.i_2,\cdots,i_n} \text{Tr}\left(|i_1, i_2, \cdots, i_n\rangle\langle i_2, i_3, \cdots, i_n, i_1|\right) = d$$

The notation $\text{dist}(\sigma, \tau)$ means the number of transpositions required to get from σ to τ. The identity $\text{dist}(\sigma, \tau) = n - \#\text{ cycles}(\sigma^{-1}\tau)$ follows from this definition.

Finally, as a remark, the matrix $K^\dagger K$ is often called the Gram matrix.

22.2 $n = 2$ case

Let us construct the Gram matrix when $n = 2$ which is $G_{\sigma,\tau} = \langle\Phi_\sigma|\Phi_\tau\rangle$. This matrix looks like

$$G = \begin{bmatrix} 1 & 1/d \\ 1/d & 1 \end{bmatrix} = \left(1 + \frac{1}{d}\right)|+\rangle\langle+| + \left(1 - \frac{1}{d}\right)|-\rangle\langle-|.$$

The Weingarten function is the inverse of this matrix, or

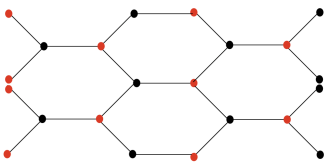
$$\text{Wg} = \frac{d^2}{d^2-1}\begin{bmatrix} 1 & -1/d \\ -1/d & 1 \end{bmatrix} = \frac{d}{d+1}|+\rangle\langle+| + \frac{d}{d-1}|-\rangle\langle-|.$$

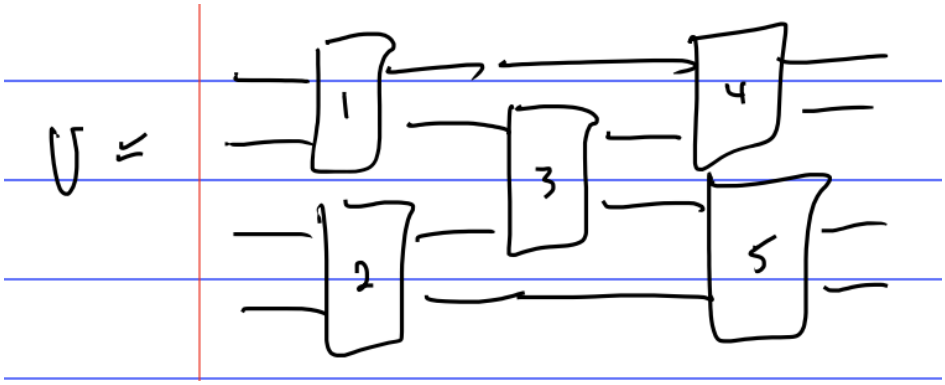
Because the off-diagonal elements are negative, we encounter a sign problem, which we need to resolve in order to write a random circuit in terms of the partition function of a statistical mechanical model.

To see how to correct the sign problem, consider the example circuit shown in the handwritten figure below

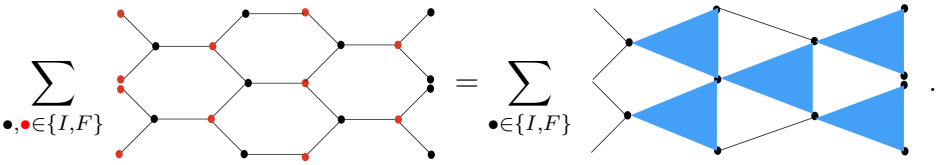
We may write the expression

$$\mathbb{E}(U \otimes U^*)^{\otimes 2} = \sum_{\bullet \in \{I,F\}}$$

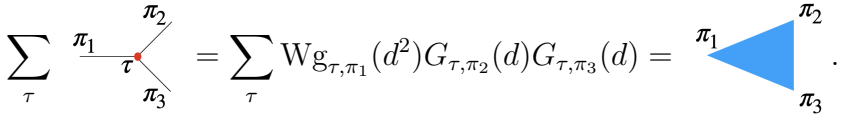




using the diagrammatic approach introduced in the last lecture. Now, we perform a step called decimation, whereby we sum over the red dots only. Remarkably, this allows us to get rid of the sign problem.



In this new notation, we have defined



We can split the evaluation up into several cases, corresponding to whether the permutations π_i are I or F. Let us denote the weight of the triangle diagram $D_{\vec{\pi}}$, where $\vec{\pi}$ is a vector of the three permutations. Then, we have:

$$(\pi_1, \pi_2, \pi_3) \in \{(I, I, I), (F, F, F)\} \to D_{\vec{\pi}} = \frac{d^4}{d^4 - 1}\left(1 - \frac{1}{d^2} \cdot \frac{1}{d} \cdot \frac{1}{d}\right) = 1,$$

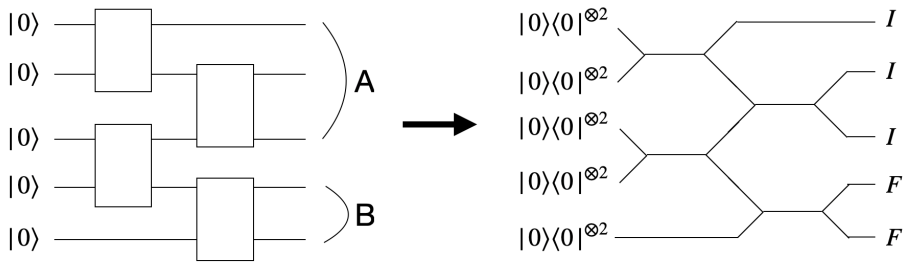
as well as

$$(\pi_1, \pi_2, \pi_3) \in \{(I, F, F), (F, I, I)\} \to D_{\vec{\pi}} = \frac{d^4}{d^4 - 1}\left(\frac{1}{d^2} - \frac{1}{d} \cdot \frac{1}{d}\right) = 0.$$

and

$$(\pi_1, \pi_2, \pi_3) \in \{(I, I, F), (I, F, I), (F, F, I), (F, I, F)\} \to D_{\vec{\pi}} = \frac{d^4}{d^4 - 1}\left(\frac{1}{d} - \frac{1}{d^3} \cdot \frac{1}{d}\right) \sim \frac{1}{d}.$$

Notice that all of these weights are positive. Therefore, it suffices to write the random circuit amplitude as a sum over all possible assignments of permutations on the black dots with the weight of a particular assignment being the product of the corresponding weights of all of the triangles. This can be written as the partition function of some effective spin model.



22.3 Estimating entanglement

As an application, we can use these mappings as a tool for understanding entanglement across a cut in a random unitary circuit. Consider the following circuit above and its dual mapping; the regions A and B are subsystems of the output density matrix.

We are interested in computing the quantity

$$Z = \mathbb{E}[\mathrm{Tr}(\rho_B^2)],$$

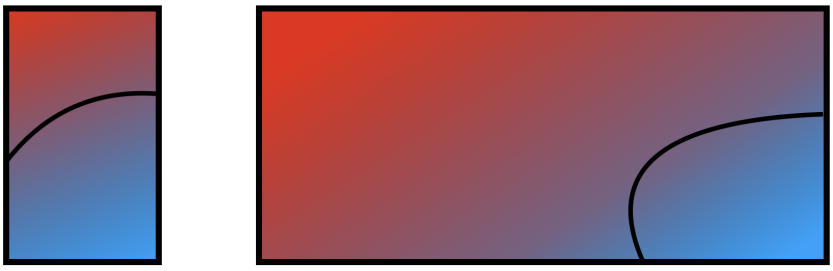
which is exactly equal to the types of expectation values we were considering. To understand how to calculate this using the statistical mechanical model formalism, we first note that in the triangle diagrams, if the two of the π's on the right side of the triangle are the same, then the π on the left corner must be equal to the π's on the right. If the two π's disagree, then the π on the left can take either of the two values with weight $\sim 1/d$. The latter case corresponds to the weight of creating a single domain wall within the triangle (here, a domain wail refers to a cut for which permutations are given different assignments on either side of the cut).

Thus, since the nodes on the right end are forced to be I in region A and F in region B, the dominant assignment of permutations to the internal nodes (i.e. the saddle point value of the partition function) occurs when one draws a domain wall that cuts the circuit into a red and blue region.

There are two limits of interest. Call the depth of the circuit t. When $t < n$, then the domain wall does not form completely and so the dominant configuration corresponds to the domain wall cutting across from the left to the right side of the circuit. The weight of this configuration scales like $(1/d)^t$, so

$$Z = \mathbb{E}[\mathrm{Tr}(\rho_A^2)] = \mathbb{E}\left[2^{-S_2(\rho_A)}\right] \approx \exp\left(t \log \frac{d^2+1}{2d}\right),$$

which is sensible because the circuit has not fully entangled the initial density matrix, so by increasing the depth by one unit, the entropy should increase by roughly $\log d$. When $t > n$, then the circuit is fully scrambled and the domain wall will stretch



from the upper/lower ends of the circuit to the right side. In this case $Z \approx d^{-n/2}$ or $S_2 = \frac{n}{2} \log d$ which implies that the entanglement has saturated. These cases are visually depicted in the figure above.

See section IV.A of 1804.09737 (originally 1608.06950) for more information.

Recent work in this area includes the following:

- Unitary + measurement circuits: Here, one alternates between applying columns of random unitaries in a brickwork architecture with random measurements inserted between adjacent columns. Here, a phase transition from a volume law (maximally entangled) phase to an area law (product state) phase is found as a function of the rate of measurement
- Random tensors and random unitary circuits are candidates for establishing quantum supremacy
- Computing S_k for $k \geq 2$ and analytic continuation to $k = 1$ using the replica trick

22.4 Monogamy of entanglement

A simple definition of monogamy of entanglement is that if Alice and Bob are maximally correlated, then neither of them can be correlated with a third party. We will better try to understand this concept through two methods:

- Using symmetry (de Finetti theorems)
- Using information theory (approximate Markov states)

A motivating example is mean field theory. We start out with the Hamiltonian

$$\mathcal{H} = \sum_{i \sim j} h_{ij}$$

where h are local Hamiltonians and $i \sim j$ means that i and j are neighbors. We will next make a mean field approximation where we assume that the Hamiltonian is close enough to one in which each particle interacts evenly among all other neighbors:

$$\mathcal{H} \approx \mathcal{H}_{MF} = \frac{D}{n} \sum_{1 \leq i < j \leq n} h_{ij}$$

where D is the number of neighbors. For example, if $h_{ij} = F_{ij}$, then the ground state is the singlet state $\frac{|01\rangle - |10\rangle}{\sqrt{2}}$. If $n = 3$, we call the system frustrated because we cannot have all pairs of particles forming singlets with each other.

We claim that the ground state of $\mathcal{H}_{MF}$ look like $\rho^{\otimes n}$.

To see this, we note that $[\mathcal{H}_{MF}, P_d(\pi)] = 0$ for all $\pi \in S_n$, since the mean field Hamiltonian is invariant under swapping the particles. Therefore, we may write

$$\mathrm{Tr}(\mathcal{H}_{MF} \psi_{gs}) = \mathrm{Tr}\left(\mathcal{H}_{MF} \frac{1}{n!} \sum_{\pi \in S_n} P_d(\pi) \psi_{gs} P_d(\pi)^\dagger \right).$$

Now define

$$\omega = \frac{1}{n!} \sum_{\pi \in S_n} P_d(\pi) \psi_{gs} P_d(\pi)^\dagger.$$

Thus, $[\omega, P_d(\pi)] = 0$ and thus we can choose the ground state to be symmetric WLOG. This does not prove that it must be a tensor power state, and we will finish the proof next time.

Why not use the pure state

$$|\psi\rangle \propto \sum_{\pi \in S_n} P_d(\pi) |\psi_{gs}\rangle,$$

which is also a valid eigenstate that is symmetric? The problem is that this can be zero, so it is safer to use density matrices over quantum states.

The de Finetti theorem states that

$$\omega_{ij} = \int d\mu(\rho)\, \rho^{\otimes 2}.$$

Why do we need a mixture (the integral)? An example would be the cat state $(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})/\sqrt{2}$, which is not close to a product state and can only be expressed as a mixture.

The original classical version of de Finetti's theorem was proved in 1931 by Bruno de Finetti. It states that suppose p is an infinitely exchangeable probability distribution, or

$$p(x_1, x_2, \cdots) = p(x_{\pi(1)}, x_{\pi(2)}, \cdots) \, \forall \pi.$$

Then $\exists \mu$ such that $\forall k$

$$p(x_1, x_2, \cdots, x_k) = \int d\mu(q) \, q(x_1) \cdots q(x_k),$$

or equivalently $p_{1\ldots k} = \int d\mu(q) \, q^{\otimes k}$. So naively, permutation invariance is not the same as independence, because we can allow for mixtures of IID distributions and still preserve permutation invariance. In 1980, Diaconis and Freedman made this result more quantitative. In particular, they found that if

$$p(x_1, \cdots, x_n) = p(x_{\pi(1)}, \cdots, x_{\pi(n)}) \, \forall n,$$

then

$$\frac{1}{2} \| \, p_{1\ldots k} - \int d\mu(q) \, q^{\otimes k} \, \|_1 \leq \min \left(\frac{k(k-1)}{2n}, \frac{k|x|}{n} \right).$$

In 2002, Caves Fuchs, and Schack derived a quantum version of de Finetti's theorem. We will follow the treatment in Chiribella (2010), 1010.1875.

We first purify $\rho_{A_1,\ldots,A_n} \to |\psi\rangle_{A_1,B_1,\ldots,A_n,B_n} \in \mathrm{Sym}^n \, \mathbb{C}^{d^2}$. Thus, it suffices to prove the de Finetti theorem for pure states.