Q. Inf. Science 3 (8.372 / 18.S996) — Fall 2022

Assignment 5

Due: **Friday**, *Oct 14, 2022 at* ***5pm*** *on gradescope*

1. **Converse theorem for quantum compression**
   Consider a quantum data compression protocol for $\rho^{\otimes n}$ with the following components.

   (a) Alice starts with the A part of $|\phi_\rho\rangle_{RA}^{\otimes n}$

   (b) Alice applies an encoding map $\mathcal{E}$ to A, creating output Q. Let $\dim Q =: M$.

   (c) Alice sends Q to Bob.

   (d) Bob applies a decoding map $\mathcal{D}$ to Q, producing output B.

   Let $\mathcal{E}$ and $\mathcal{D}$ have Kraus operators $\{E_k\}$ and $\{D_l\}$ respectively. The fidelity of the compression scheme is

   $$f := F(\phi_\rho^{\otimes n}, \mathcal{D}(\mathcal{E}(\phi_\rho^{\otimes n}))), \tag{1}$$

   where fidelity is defined in the usual way as $F(\alpha, \beta) = \left\|\sqrt{\alpha}\sqrt{\beta}\right\|_1$.

   (a) Show that

   $$f^2 = \sum_{k,l} \left|\operatorname{tr}\left(D_l E_k \rho^{\otimes n}\right)\right|^2. \tag{2}$$

   (b) Show that $\operatorname{rank} D_l E_k \leq M$.

   (c) Show that

   $$f^2 \leq \sum_{k,l} \operatorname{tr}\left[D_l E_k \rho^{\otimes n} E_k^\dagger D_l^\dagger\right] \operatorname{tr}\left[P_{kl}\rho^{\otimes n}\right], \tag{3}$$

   where each P_{kl} is a projector of rank $\leq M$. [Hint: The matrix Cauchy-Schwarz inequality states that $\operatorname{tr}\left[A^\dagger B\right] \leq \sqrt{\operatorname{tr}[A^\dagger A] \operatorname{tr}[B^\dagger B]}$ and may be helpful.]

   (d) Show that if $M = 2^{nR}$ for $R < S(\rho)$ then f approaches 0 exponentially quickly as $n \to \infty$.