Q. Inf. Science 3 (8.372 / 18.S996) — Fall 2022

Assignment 7

Due: **Friday**, *Oct 28, 2022 at* ***5pm***

1. **Unital channels** Let $\mathcal{N}$ be a completely positive (cp) map whose input and output both are d-dimensional. We say $\mathcal{N}$ is *unital* if $\mathcal{N}(I) = I$. Recall also that $\mathcal{N}$ is trace preserving (tp) if $\text{tr}[\mathcal{N}(X)] = \text{tr}[X]$ for any X.

   (a) Show that $\mathcal{N}$ is unital if and only if $\mathcal{N}^\dagger$ is trace preserving.

   (b) A channel is *mixed unitary* if it can be written as $\mathcal{N}(X) = \sum_{i=1}^m p_i U_i X U_i^\dagger$ with $p_1, \ldots, p_m$ a probability distribution and $U_1, \ldots, U_m$ unitaries. Show that all mixed unitary channels are unital.

   Remark: Stochastic matrices have nonnegative entries and their columns each sum to 1. These are the classical analogue of channels. Doubly-stochastic matrices also have their rows summing to 1. The Birkhoff-von Neumann theorem states that doubly-stochastic matrices are mixtures of permutations (which could be said to be the classical analogue of unitary matries.) So one might guess that unital channels are always mixtures of unitaries. This is true when $d = 2$ but false when $d > 3$; a counter-example is the *Werner-Holevo* channel that we will discuss below.

   (c) *[Optional.]* Prove that when $d = 2$ all unital channels are mixed unitary. As a hint, first argue that if $\mathcal{N}$ has single qubit input and output then it has the form $\mathcal{N}(\frac{I+\vec{x}\cdot\vec{\sigma}}{2}) = \frac{I+(A\vec{x}+\vec{b})\cdot\vec{\sigma}}{2}$ for some matrix A and vector $\vec{b}$. We use the notation $\vec{x}\cdot\sigma$ to mean $x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$ where σ_i are the Pauli matrices.

   (d) Prove that a channel $\mathcal{N}$ is unital if and only if it is entropy non-decreasing, that is, $S(\mathcal{N}(\rho)) \geq S(\rho)$ for all ρ.

   (e) Let H be a Hermitian matrix, T a positive real number, and $\sigma := e^{-H/T} / \text{tr}\left[e^{-H/T}\right]$. Suppose $\mathcal{N}(\sigma) = \sigma$. For any ρ define its *free energy* to be

   $$F(\rho) := \text{tr}[H\rho] - TS(\rho). \tag{1}$$

   Show that $F(\mathcal{N}(\rho)) \leq F(\rho)$ for any ρ.

   (f) *[Optional.]* If $\mathcal{N}$ is a unital channel then prove that the eigenvalues of $\mathcal{N}(\rho)$ are majorized by those of ρ. (This means that $\alpha_1 + \cdots + \alpha_k \leq \beta_1 + \cdots + \beta_k$ for any k where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d$ are the sorted eigenvalues of $\mathcal{N}(\rho)$ and $\beta_1 \geq \beta_2 \geq \cdots \beta_d$ are the sorted eigenvalues of ρ.)

2. **Entanglement-assisted quantum capacity.**

   (a) *Quantum capacity.* Denote the entanglement-assisted capacity of a quantum channel for sending qubits (resp. cbits) by Q_E (resp. C_E). Relate Q_E to C_E using teleportation and super-dense coding.

   (b) *Classical channels.* The entanglement-assisted capacity theorem states that $C_E(\mathcal{N}) = \max_\tau I(A:B)_\tau$ (see pset 6 for definition of τ). Consider the special case of a classical channel $\mathcal{N}(\rho) := \sum_{x,y} \langle x|\,\rho\,|x\rangle\, N(y|x)\,|y\rangle\!\langle y|$ and show that $C_E(\mathcal{N}) = C(N)$. In other words, show that entanglement doesn't increase the capacity of classical channels, and that Shannon's noisy coding theorem can recovered as a special case of the entanglement-assisted capacity theorem.

   (c) *Depolarizing channel.* Let $\mathcal{D}_p^d$ (abbreviated $\mathcal{D}$) denote the depolarizing channel on d dimensions with depolarization probability p, defined as

   $$\mathcal{D}_p^d(\rho) = (1-p)\rho + p\frac{I}{d}, \tag{2}$$

   for ρ a d-dimensional density matrix. Observe that $\mathcal{D}(U\rho U^\dagger) = U\mathcal{D}(\rho)U^\dagger$ for any unitary U. Use this property and the input concavity property of C_E to show that $C_E(\mathcal{D}_p^d, \rho)$ is maximized for $\rho = I/d$. Calculate $C_E(\mathcal{D}_p^d, I/d)$.

   (d) *Enhancement from entanglement.* The classical capacity of the depolarizing channel $C(\mathcal{D})$ can be shown to be maximized by applying the HSW theorem to the ensemble where each of the basis states $|1\rangle, \ldots, |d\rangle$ appears with probability $1/d$. Calculate $C(\mathcal{D}_p^d)$. What is the ratio $\frac{C_E(\mathcal{D})}{C(\mathcal{D})}$ in the limits $p \to 0$ and $p \to 1$ as a function of d?

3. **Additivity violation of $S_\infty^{\min}$ with the Werner-Holevo channel.** In this problem we will see a simple version of the sort of additivity violation that is possible with entangled inputs.

   (a) First, we relate additivity violations of χ to those of a related quantity. For a channel $\mathcal{N}$, its *minimum output entropy* is defined to be

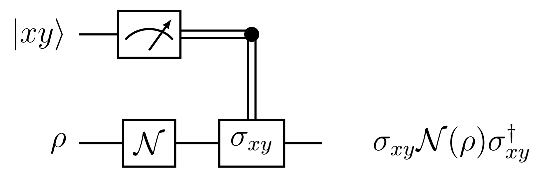
   $$S^{\min}(\mathcal{N}) := \min_\rho S(\mathcal{N}(\rho)). \tag{3}$$

   Show that

   $$S^{\min}(\mathcal{N}_1 \otimes \mathcal{N}_2) \le S^{\min}(\mathcal{N}_1) + S^{\min}(\mathcal{N}_2) \tag{4}$$

   for any channels $\mathcal{N}_1, \mathcal{N}_2$. An additivity violation of $S^{\min}$ occurs when this inequality is strict.

   (b) Given a channel $\mathcal{N}$ that maps $\mathbb{C}^{d_A}$ to $\mathbb{C}^{d_B}$, define a new channel $\mathcal{N}'$ with a d_B^3-dimensional input and d_B-dimensional output as follows. The input is interpreted as a pair of registers: one with dimension d_B^2 and another with dimension d_B. The channel measures the first register and obtains outcomes $x, y \in [d_B]$. It then applies $\mathcal{N}$ to the second register and finally σ_{xy} (i.e. the generalized Pauli matrix; see pset 6 for definition). Bob receives output $\sigma_{xy}\mathcal{N}(\rho)\sigma_{xy}^\dagger$ where ρ is contents of the second register. This is seen in the following quantum circuit.



Prove that

$$\chi(\mathcal{N}') = \log(d_B) - S^{\min}(\mathcal{N}). \tag{5}$$

In this way, we can relate properties of $S^{\min}$, such as [non-]additivity, to properties of χ.

As a reminder, here is the definition of χ. $\chi(\mathcal{N}')$ is the maximum over all probability distributions $p = (p_1, \ldots, p_m)$ and d_B^3-dimensional states $\sigma_1, \ldots, \sigma_m$ of

$$S(\sum_i p_i \mathcal{N}'(\sigma_i)) - \sum_i p_i S(\mathcal{N}'(\sigma_i)). \tag{6}$$

(c) Demonstrating additivity violations for $S^{\min}$ is challenging.[1] Instead we will study a related quantity, called $S^{\min}_\infty$. Define $S_\infty(\rho) = -\log \|\rho\|$ where $\|\rho\|$ is the largest singular value of ρ. (This is a special case of the Rényi entropy $S_\alpha(\rho) := \frac{1}{1-\alpha}\log \operatorname{tr} \rho^\alpha$.) Define

$$S^{\min}_\infty(\mathcal{N}) := \min_\rho S_\infty(\mathcal{N}(\rho)) = -\log \max_\rho \|\mathcal{N}(\rho)\|. \tag{7}$$

Let $\mathcal{N}$ be the Werner-Holevo channel with d-dimensional inputs and outputs:

$$\mathcal{N}(\rho) := \frac{I - \rho^T}{d-1}. \tag{8}$$

Here I is the identity matrix and ρ^T denotes the transpose operation, i.e. $\rho^T_{i,j} = \rho_{j,i}$. At first it is not even obvious that this is a valid quantum channel. Prove that it is a tpcp map by showing that the Choi-Jamiolkowski state $(\mathcal{N} \otimes \mathrm{id})(\Phi)$ is a valid quantum state. Here Φ is the maximally entangled state. As a hint, recall that the partial transpose of Φ is proportional to the swap operator.

For the rest of the pset, we use $\mathcal{N}$ to refer to this Werner-Holevo channel.

(d) Calculate $S^{\min}_\infty(\mathcal{N})$. As a hint, show that the minimum is achieved on a pure input state.

(e) Calculate $S_\infty((\mathcal{N} \otimes \mathcal{N})(\Phi))$. Evaluate your expression when $d = 3$. How does this quantity compare with $2S^{\min}_\infty(\mathcal{N})$? What can you conclude about the additivity of $S^{\min}_\infty$?

[1]This was first achieved by Hastings and later simplified somewhat by Aubrun, Szarek and Werner. These references are included for completeness but you do not need to look at them to solve the current problem.