Q. Inf. Science 3 (8.372) — Fall 2024

Assignment 10

Due: **Tuesday**, *Dec 3, 2024 at* ***9pm***

Turning in your solutions: Upload a single pdf file to gradescope.

1. **A pretty good problem**

   In class it was claimed that $I(A;C|B)_\rho = 0$ if and only if

$$\rho^{ABC} = \sum_\alpha p_\alpha \sigma_\alpha^{AB_\alpha^L} \otimes \omega_\alpha^{B_\alpha^R C}, \tag{1}$$

   where the system B can be decomposed as

$$B = \bigoplus_\alpha B_\alpha^L \otimes B_\alpha^R \tag{2}$$

   (a) Assume that $\rho > 0$, i.e. ρ is full rank. Show that eq. (1) implies that $\rho = e^{-X_{AB}-Y_{BC}}$ for some commuting Hermitian operators X_{AB}, Y_{BC}. (The reverse implication is also true but you don't need to prove it here.)

   (b) **Adjoint channels**. Define the *Hilbert-Schmidt* inner product between two matrices to be

$$\langle X, Y\rangle := \mathrm{tr}\big[X^\dagger Y\big]. \tag{3}$$

   The adjoint of a superoperator $T \in L(L(A), L(B))$ with respect to this inner product is defined by the expression

$$\langle X, T(Y)\rangle = \langle T^\dagger(X), Y\rangle. \tag{4}$$

   This is also known as the Heisenberg picture for quantum operations.

   i. If $T(\rho) = \sum_{i\in[k]} A_i \rho A_i^\dagger$ then what are the Kraus operators of $T^\dagger$?

   ii. Let T be a superoperator, not necessarily a valid quantum operation. What condition on $T^\dagger$ is equivalent to the condition that T is trace preserving? What condition on $T^\dagger$ is equivalent to the condition that T is completely positive?

   iii. tr_C is a quantum channel from $B \otimes C$ to B. What is $\mathrm{tr}_C^\dagger$?

   iv. Let $\mathcal{M} = \{M_1, \ldots, M_k\}$ be a POVM. Define a new POVM $\mathcal{M}\circ\mathcal{N}$ by applying $\mathcal{N}$ and then measuring $\mathcal{M}$. Write down the POVM elements of $\mathcal{M}\circ\mathcal{N}$ and justify your answer.

(c) The *Petz recovery map*, also known as the transpose channel, is a method of approximately reversing a quantum channel $\mathcal{N}$ with respect to input state σ. Given $\sigma, \mathcal{N}$ the Petz map is the operation $\mathcal{P}_{\sigma,\mathcal{N}}$ defined as

$$\mathcal{P}_{\sigma,\mathcal{N}}(\rho)\sigma^{1/2}\mathcal{N}^\dagger(\mathcal{N}(\sigma)^{-1/2}\rho\mathcal{N}(\sigma)^{-1/2})\sigma^{1/2} \tag{5}$$

Assume for convenience that σ and $\mathcal{N}(\rho)$ are full rank. Show that $\mathcal{P}_{\sigma,\mathcal{N}}(\rho)$ is a TPCP map, i.e. a valid quantum operation.

(d) Calculate the Petz recovery map for a state σ_{BC} and $\mathcal{N} = \mathrm{tr}_C$; denote this $\mathcal{P}$. Show that if σ_{ABC} satisfies $I(A;C|B)_\sigma = 0$ then $(\mathrm{id}_A \otimes \mathcal{P}_{B\to BC})\sigma_{AB} = \sigma_{ABC}$.

(e) Now apply the Petz recovery map to the state distinguishability problem. Here we are given state ρ_x with probability p_x (for $x \in [n]$) and want to guess x. Let

$$\sigma^{XQ} = \sum_x p_x \left|x\middle\rangle\!\middle\langle x\right|^X \otimes \rho_x^Q, \tag{6}$$

and calculate the Petz recovery map for the operation tr_X. Interpret this as a measurement with operators $M_x = \rho^{-1/2}A_x\rho^{-1/2}$, with $\rho := \sigma^Q = \sum_x p_x\rho_x$ and find A_x. This measurement is called the *pretty good measurement*, or PGM.

(f) With the same setup as the previous part, let $\{N_x\}$ be the measurement which achieves the optimal guessing probability, i.e. it achieves the maximum in

$$P_{\mathrm{opt}} := \max_{\{N_x\}} \sum_x p_x \,\mathrm{tr}[N_x\rho_x] \tag{7}$$

Let $P_{\mathrm{PGM}} := \sum_x p_x \,\mathrm{tr}[M_x\rho_x]$ where $\{M_x\}$ is the PGM from the previous part. Prove that

$$P_{\mathrm{PGM}} \geq P_{\mathrm{opt}}^2 \tag{8}$$

This justifies the term "pretty good measurement."

2. **Rényi subaddivity?**

The error term in merging is proportional to:

$$\mathrm{tr}\left[\psi_{AR}^2\right] + \mathrm{tr}\left[\psi_A^2\right]\mathrm{tr}\left[\psi_R^2\right] = 2^{-S_2(AR)} + 2^{-S_2(A)-S_2(R)}. \tag{9}$$

If it were true that one of these terms was dominated by the other, then we could give a simplified upper bound. This problem will explore that possibility.

(a) Show that there exists a choice of ψ for which $S_2(AR) \gg S_2(A) + S_2(R)$ and another choice where $S_2(AR) \ll S_2(A) + S_2(R)$.

(b) How does this change when we replace the Rènyi entropy S_2 with the von Neumann entropy S?

3. **An area law for the mutual information**

   Consider a local Hamiltonian $H = \sum_{(i,j)\in E} h_{i,j}$ where E is a collection of edges defining a graph over a vertex set V and $h_{i,j}$ is a term acting on the qudits at sites i and j. Partition V into $A, \bar{A}$ and write H as

$$H = H_A + H_{\bar{A}} + H_\partial, \tag{10}$$

   where H_A (resp. $H_{\bar{A}}$) are the terms acting entirely within $A, \bar{A}$ and H_∂ is the sum of all the interactions between A and $\bar{A}$, i.e. either $i \in A, j \in \bar{A}$ or vice versa.

   Let $\sigma := \frac{e^{-H/T}}{\mathrm{tr}\left[e^{-H/T}\right]}$ be the GIbbs state.

   (a) Prove that

$$I(A;\bar{A})_\sigma \leq \frac{\|H_\partial\|}{T} \tag{11}$$

   (b) What does this tell us about the amount of entanglement between A and $\bar{A}$ in the ground state? If $T = 0$ then σ is the ground state (or mixture over all ground states) but then eq. (11) is vacuous. However, suppose we further assume a bound on the density of states. Specifically, that the ground state energy is E_0 and that the number of states of energy $\leq E_0 + k$ is $\leq n^k$. Show how this yields a nontrivial bound on the ground-state entanglement entropy.

4. **Area law correction in the surface code**

   This problem is optional. But I hope you find it tempting!

   The surface (or toric) code is defined on a lattice with qubits on each edge, and stabilizer generators

$$A_s = \prod_{e\sim s} X_e \qquad\qquad B_p = \prod_{e\sim p} Z_e \tag{12}$$

   Here s refers to a "site" (or vertex) and $e \sim s$ means that e is one of the four edges touching this site. These four edges make a star. Next p refers to a "plaquette" (or square) and $e \sim p$ refers to the four edge bordering this plaquette. This is illustrated in **??**(a).

   For simplicity, we will consider the surface code with smooth boundaries, or else on the sphere, so that there are no logical qubits. This means that there is a unique state $|\psi\rangle$ stabilized by all the $\{A_s\}$ and $\{B_p\}$. (The story is similar for the case of the torus or rough boundaries when there are some logical qubits but we wish to avoid those complications for now.)

   Let A be an subregion of size $k \times k$, illustrated in **??**(b). Compute $S(\psi_A)$ as a function of k. Your answer should be of the form $\alpha k - \gamma$. The term γ is known as the topological entanglement entropy and was introduced in hep-th/0510092. (You do not need anything from that paper to solve this problem.)

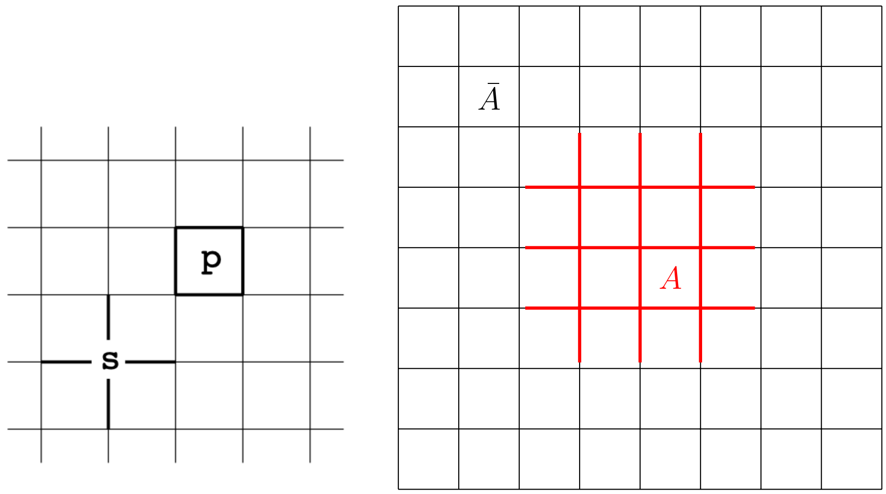

Figure 1: (a) Star A_s and plaquette B_p operators. (b) A is a subregion of size $k \times k$. Here $k = 4$.

As a hint, a stabilizer state $|S\rangle$ is defined in terms of a maximal stabilizer subgroup S of the Pauli group $P_n := \pm\{I, X, iY, Z\}^{\otimes n}$. We require that $-1 \in S$, S is abelian and $|S| = 2^n$, and then have

$$|S\rangle\langle S| = |S|^{-1} \sum_{s \in S} s = \prod_{i=1}^{n} \frac{I + g_i}{2}, \tag{13}$$

where $g_1, \ldots, g_n$ generates S. If instead $|S| = 2^m$ for $m \leq n$ then in general we obtain the mixed state

$$\rho_S := |S|^{-1} \sum_{s \in S} s = \prod_{i=1}^{m} \frac{I + g_i}{2}. \tag{14}$$

This has 2^{n-m} eigenvalues, each equal to 2^{m-n}.