# Q. Inf. Science 3 (8.372) — Fall 2024

## Assignment 3

*Due:* **Tuesday**, *Oct 1, 2024 at* ***9pm***

**Turning in your solutions:** Upload a single pdf file to gradescope.

1. **Nonnegativity of quantum relative entropy** Prove that $D(\rho\|\sigma) \geq 0$. You can use without proof the nonnegativity of the classical relative entropy. As a hint, the concavity of log means that $\sum_j p_j \log(x_j) \leq \log\left(\sum_j p_j x_j\right)$.

2. **Gibbs distributions** In this problem we define entropy with log base-$e$, i.e. ln. Also let $\exp(x) := e^x$.

   (a) Consider a classical system whose state lies in the set $\Omega$. For simplicity assume that $\Omega$ is finite. The energy is defined by function $E : \Omega \to \mathbb{R}$. The Gibbs distribution at temperature $T$ is the probability distribution

   $$g_T(x) := \frac{e^{-E(x)/T}}{Z} \qquad \text{where} \qquad Z := \sum_{x\in\Omega} e^{-E(x)/T}. \tag{1}$$

   For given $E, T$, define the [Helmholtz] free energy of a probability distribution $p$ by

   $$F(p) := \mathop{\mathbb{E}}_{x\sim p}[E(x)] - TH(p) = \sum_{x\in\Omega} p(x)[E(x) + T\ln(p(x))] \tag{2}$$

   *This paragraph is not needed to solve the problem:* As background for those unfamiliar with thermodynamics, the idea of free energy is that there might be many microscopic configurations $x$ compatible with some macroscopic variables $X$, such as pressure or volume. The probability of $X$ is then $g_T(X) = \sum_{x\in X} g_T(x)$. If every $x \in X$ has the same energy and $p$ is uniform over $X$ then $H(p) = \log|X|$ and $g_T(X) = |X|g_T(x) = e^{-F(p)/T}/Z$. Thus the free energy plays the same role as the ordinary energy when considering a distribution over macrostates.

   Prove that $g_T$ is a local minimum of the free energy, using calculus.

   (b) Calculate $F(g_T)$ and relate it to $\log Z$. Calculate $D(p\|g_T)$ and relate it to $F(p)$ and $F(g_T)$. Explain how this yields an independent proof of the result from (a).

   (c) Now we consider the quantum case. Let $H$ be a finite-dimensional Hermitian matrix. Define the Gibbs state and the free energy by

   $$\gamma_T := \frac{e^{-H/T}}{\mathrm{tr}[e^{-H/T}]} \quad \text{and} \quad F(\rho) := \mathrm{tr}[H\rho] - TS(\rho). \tag{3}$$

Prove that $\gamma_T$ is a local minimum of $F$. *Hint: Here it is easier to follow the approach of (b) than (a). To follow (b), note that* $\log(cA) = \log(A) + \log(c)I$ *if* $A$ *is a matrix and* $c$ *a scalar. If you follow (a), you might find the formula*

$$\ln(A+B) = \ln(A) + \int_0^\infty \mathrm{d}z \, \frac{1}{A+zI} B \frac{1}{A+B+zI}. \tag{4}$$

*useful to evaluate the gradient of* $F$.

(d) Is $F$ concave, convex or neither? Does this tell us anything about the relation between local and global minima of $F$?

(e) Suppose that $F(\rho) \leq F(\gamma_T) + \delta$. What can you say about the trace distance between $\rho$ and $\gamma_T$?
As a hint, the quantum Pinsker inequality states that $D(\rho\|\sigma) \geq \frac{1}{2}\|\rho - \sigma\|_1^2$.

3. **Converse theorem for quantum compression**
Consider a quantum data compression protocol for $\rho^{\otimes n}$ with the following components.

(a) Alice starts with the $A$ part of $|\phi_\rho\rangle_{RA}^{\otimes n}$
(b) Alice applies an encoding map $\mathcal{E}$ to $A$, creating output $Q$. Let $\dim Q =: M$.
(c) Alice sends $Q$ to Bob.
(d) Bob applies a decoding map $\mathcal{D}$ to $Q$, producing output $B$.

Let $\mathcal{E}$ and $\mathcal{D}$ have Kraus operators $\{E_k\}$ and $\{D_l\}$ respectively. The fidelity of the compression scheme is

$$f := F(\phi_\rho^{\otimes n}, \mathcal{D}(\mathcal{E}(\phi_\rho^{\otimes n}))), \tag{5}$$

where fidelity is defined in the usual way as $F(\alpha, \beta) = \left\|\sqrt{\alpha}\sqrt{\beta}\right\|_1$.

(a) Show that

$$f^2 = \sum_{k,l} \left|\operatorname{tr}\left(D_l E_k \rho^{\otimes n}\right)\right|^2. \tag{6}$$

(b) Show that $\operatorname{rank} D_l E_k \leq M$.

(c) Show that

$$f^2 \leq \sum_{k,l} \operatorname{tr}\left[D_l E_k \rho^{\otimes n} E_k^\dagger D_l^\dagger\right] \operatorname{tr}\left[P_{kl}\rho^{\otimes n}\right], \tag{7}$$

where each $P_{kl}$ is a projector of rank $\leq M$. [Hint: The matrix Cauchy-Schwarz inequality states that $\operatorname{tr}\left[A^\dagger B\right] \leq \sqrt{\operatorname{tr}[A^\dagger A]\operatorname{tr}[B^\dagger B]}$ and may be helpful.]

(d) Show that if $M = 2^{nR}$ for $R < S(\rho)$ then $f$ approaches 0 exponentially quickly as $n \to \infty$.