Q. Inf. Science 3 (8.372) — Fall 2024

Assignment 5

Due: **Wednesday**, *Oct 16, 2024 at* ***9pm***

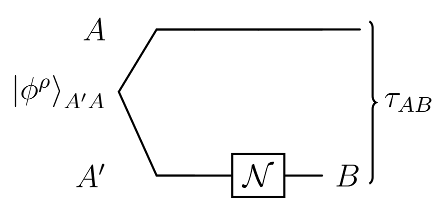
Turning in your solutions: Upload a single pdf file to gradescope.

1. **Entanglement-assisted capacity** For classical channels, shared randomness does not help the capacity. One way to see this is that feedback can be used to share randomness, and feedback does not help the capacity. But for quantum channels, we know that entanglement between sender and receiver can improve the classical capacity, as seen in the example of super-dense coding. In fact, free entanglement dramatically simplifies the quantum capacity. Let $C_E(\mathcal{N})$ denote the asymptotic rate that $\mathcal{N} : A' \to B$ can send classical bits when assisted by unlimited EPR pairs between sender and receiver. It turns out that

$$C_E(\mathcal{N}) = \max_\rho C_E(\mathcal{N}, \rho) \qquad \text{where } C_E(\mathcal{N}, \rho) := I(A : B)_\tau, \tag{1}$$

ρ is maximized over all density matrices on A', $\phi^\rho_{AA'}$ is a purification of ρ, and

$$\tau_{AB} = (\mathrm{id}_A \otimes \mathcal{N}_{A' \to B})(\phi^\rho_{AA'}). \tag{2}$$



   (a) As a warmup, we derive a decomposition of the d-dimensional depolarizing channel. Define the generalized Paulis (also called Weyl-Heisenberg operators) by

$$\sigma_{xy} := \sum_{z=0}^{d-1} \omega^{zy} |z + x\rangle \langle z| , \tag{3}$$

   where $x, y \in \{0, 1, \ldots, d-1\}$, $z + x$ is defined mod d and $\omega := e^{2\pi i/d}$. Show that

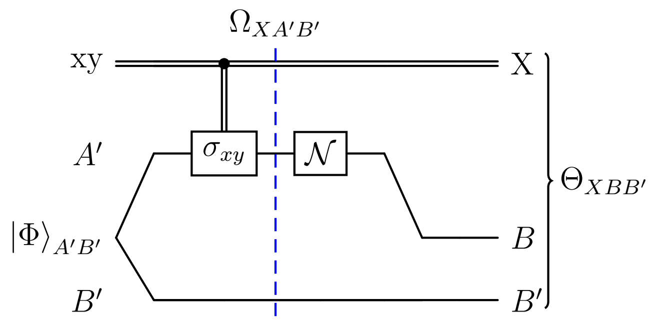
$$\mathcal{E}(M) := \frac{1}{d^2} \sum_{x,y} \sigma_{x,y} M \sigma^\dagger_{x,y} = \frac{I}{d} \mathrm{tr}[M], \tag{4}$$

   for any matrix M.

   (b) Now return to entanglement-assisted capacity. Consider the special case in which the maximum in eq. (1) is achieved by $\rho = I/d$, where $d = |A|$. Consider the following coding scheme for Alice. She chooses x, y uniformly randomly, applies σ_{xy} to her half of an entangled state

$$|\Phi\rangle_{A'B'} := \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle_{A'} \otimes |i\rangle_{B'} \tag{5}$$

and then sends system A' through the channel. We can express the resulting ensemble as a single state with system X containing Alice's encoding and systems B and B' representing Bob's channel output and piece of the shared entanglement. This is depicted in the following circuit diagram.



$$\Omega_{XA'B'} := \frac{1}{d^2} \sum_{xy} |xy\rangle\langle xy|_X \otimes (\sigma_{xy} \otimes I)\Phi_{A'B'}(\sigma_{xy} \otimes I)^\dagger \tag{6}$$

$$\Theta_{XBB'} := (\mathcal{N}_{A'\to B} \otimes \mathrm{id}_{B'X})(\Omega) \tag{7}$$

Compute $I(X : BB')_\Theta$ in terms of $I(A : B)_\tau$. Using the HSW theorem, what can you then conclude about C_E? [Hint: Recall that $(X \otimes I)|\Phi\rangle = (I \otimes X^T)|\Phi\rangle$.]

(c) *Input concavity.* Show that $C_E(\mathcal{N}, \rho)$ is independent of the choice of purification ϕ^ρ. Show that $C_E(\mathcal{N}, \rho)$ is concave in the input ρ. [Hint: purify $\sum_x p(x)|x\rangle\langle x| \otimes \phi^{\rho_x}$.]

(d) [Optional.] Assume now that eq. (1) has been shown to be true. Prove that the capacity is additive, i.e. that

$$C_E(\mathcal{N}_1 \otimes \mathcal{N}_2) = C_E(\mathcal{N}_1) + C_E(\mathcal{N}_2). \tag{8}$$

2. **Information Theory for Quantum Metrology**

In quantum metrology/sensing, we are given a state from a parameterized family ρ_θ and want to estimate θ. We consider the simplest case here where $\theta \in \mathbb{R}$. In this problem we will study Quantum Cramér-Rao bound and use it to show that, when a signal to detect is encoded in a product state, one cannot improve the sensitivity of detecting the signal beyond the standard quantum limit (QSL) scaling in the number of particles, even if joint measurements are performed. In other words, in order to go beyond SQL, sensing particles must be entangled or correlated before or during sensing signal is accumulated.

Specifically, consider a n-particle product state, not necessarily a tensor power:

$$\rho_\theta^{(n)} = \rho_{1,\theta} \otimes \rho_{2,\theta} \otimes \cdots \otimes \rho_{n,\theta}. \tag{9}$$

Each factor $\rho_{k,\theta}$ depends on the parameter θ. We consider performing the measurement of an n-particle joint observable $A^{(n)}$ that will serve as our locally unbiased estimator for θ near $\theta = 0$:

$$\langle A^{(n)} \rangle_\theta \equiv \mathrm{Tr}\{\rho_\theta^{(n)} A^{(n)}\} = \theta + \mathcal{O}(\theta^2). \tag{10}$$

Our goal is to prove the Quantum Cramér-Rao bound

$$\mathrm{Var}(A^{(n)}) = \mathrm{Tr}\{\rho_0^{(n)} (A^{(n)})^2\} \geq \frac{1}{\sum_k F_{Q,k}}, \tag{11}$$

where $F_{Q,k}$ is the quantum Fisher information for the state $\rho_{k,\theta}$. To solve this problem, we will first derive an explicit formula for $F_{Q,k}$.

Symmetric Logarithmic Derivative

(a) Assume we a given a full-rank density matrix $\rho > 0$. We define superoperators associated with ρ:

$$\mathrm{Mult}_\rho[A] \equiv \frac{1}{2}(\rho A + A\rho) \tag{12}$$

$$\mathrm{Divi}_\rho[A] \equiv \mathrm{Mult}_\rho^{-1}[A]. \tag{13}$$

Prove that, for full-rank (i.e. positive definite) ρ, Mult_ρ is invertible and hence Divi_ρ is well defined.

(b) The implicit definition of the symmetric logarithmic derivative (SLD) is

$$\partial_\theta \rho_\theta = \frac{1}{2}(\rho_\theta L + L\rho_\theta), \tag{14}$$

where L depends on ρ_θ (and its dependence on θ). Show that L can be written as

$$L = 2\sum_{jk} \frac{(\partial_\theta \rho_\theta)_{jk}}{\lambda_j + \lambda_k} |j\rangle\langle k|, \tag{15}$$

where $\lambda_j > 0$ and $|j\rangle$ are eigenvalues and eigenvectors of ρ, and $(\partial_\theta \rho_\theta)_{jk}$ are the matrix elements of the derivative of ρ, in this basis. Show that L is Hermitian.

Quantum Fisher Information

In this part, we explore how quantum relative entropy $D(\rho_\theta||\rho_0)$ and quantum Fisher information F_Q can be related to one another. Recall the definitions

$$D(\rho_\theta||\rho_0) = \mathrm{Tr}\{\rho_\theta \log \rho_\theta - \rho_\theta \log \rho_0\} \tag{16}$$
$$F_Q = \mathrm{Tr}\{\rho_0 L^2\}, \tag{17}$$

where L is the symmetric logarithmic derivative. Directly evaluating the second order derivative of $D(\rho_\theta||\rho_0)$ in θ does *not* lead to F_Q, but instead results in a different information quantity. This discrepancy originates from evaluating the logarithmic derivative, which we explore here.

(c) Re-express $D(\rho_\theta||\rho_0)$ in terms of $\rho_\theta, \partial_\theta\rho_\theta, \partial_\theta \log \rho_\theta$ and their derivatives. Do not evaluate $\partial_\theta \log \rho_\theta$, but instead replace it with L. Show that replacing $\partial_\theta \log \rho_\theta$ with L turns $\partial_\theta^2 D(\rho_\theta||\rho_0)$ into F_Q.

(Hint: evaluate $\mathrm{Tr}\{\rho L\}$ and $\partial_\theta \mathrm{Tr}\{\rho L\}$, which will be useful to simplify your expressions.)

Additivity

(d) Show the additivity of quantum Fisher information:

$$F_Q(\rho_\theta \otimes \sigma_\theta) = F_Q(\rho_\theta) + F_Q(\sigma_\theta). \tag{18}$$

Quantum Cramér-Rao bound

(e) Complete your proof to obtain the desired quantum Cramér-Rao bound for product states. What is the optimal signal-to-noise ratio (SNR) for any protocol based on measuring n-particle observable $A^{(n)}$ for n uncorrelated particles, and how is it bounded? Here SNR means the ratio of expectation value to standard deviation. What if the *measurement* protocol is adaptive: measurement outcomes performed on $\rho_{1,\theta}, \ldots, \rho_{k,\theta}$ in earlier steps determine the choice of POVM to perform on $\rho_{k+1,\theta}$ in later steps? Relate this result to any single-particle sensing protocol that is sequentially repeated n times?