Q. Inf. Science 3 (8.372) — Fall 2024

Assignment 6

Due: **Tuesday**, *Oct 22, 2024 at* ***9pm***

Turning in your solutions: Upload a single pdf file to gradescope.

1. **Entanglement-assisted quantum capacity.**

   (a) *Quantum capacity.* Denote the entanglement-assisted capacity of a quantum channel for sending qubits (resp. cbits) by Q_E (resp. C_E). Relate Q_E to C_E using teleportation and super-dense coding.

   (b) *Depolarizing channel.* Let $\mathcal{D}_p^d$ (abbreviated $\mathcal{D}$) denote the depolarizing channel on d dimensions with depolarization probability p, defined as

   $$\mathcal{D}_p^d(\rho) = (1-p)\rho + p\frac{I}{d}, \tag{1}$$

   for ρ a d-dimensional density matrix. Observe that $\mathcal{D}(U\rho U^\dagger) = U\mathcal{D}(\rho)U^\dagger$ for any unitary U. Use this property and the input concavity property of C_E to show that $C_E(\mathcal{D}_p^d, \rho)$ is maximized for $\rho = I/d$. Calculate $C_E(\mathcal{D}_p^d, I/d)$.

   (c) *Enhancement from entanglement.* The classical capacity of the depolarizing channel $C(\mathcal{D})$ can be shown to be maximized by applying the HSW theorem to the ensemble where each of the basis states $|1\rangle, \ldots, |d\rangle$ appears with probability $1/d$. Calculate $C(\mathcal{D}_p^d)$. What is the ratio $\frac{C_E(\mathcal{D})}{C(\mathcal{D})}$ in the limits $p \to 0$ and $p \to 1$ as a function of d?

   (d) *[Optional:] Classical channels.* The entanglement-assisted capacity theorem states that $C_E(\mathcal{N}) = \max_\tau I(A:B)_\tau$ (see previous pset for definition of τ). Consider the special case of a classical channel $\mathcal{N}(\rho) := \sum_{x,y} \langle x|\rho|x\rangle N(y|x)\,|y\rangle\langle y|$ and show that $C_E(\mathcal{N}) = C(N)$. In other words, show that entanglement doesn't increase the capacity of classical channels, and that Shannon's noisy coding theorem can recovered as a special case of the entanglement-assisted capacity theorem.

   This problem is not optional because it's particularly hard, just because I want to keep the pset length down.

2. **Chernoff bound and Pinsker inequality**. In this problem you will derive the quantum Pinsker inequality and explore some applications.

   The Pinsker inequality is
   $$D(\rho\|\sigma) \geq \frac{1}{2\ln 2}\|\rho - \sigma\|_1^2. \tag{2}$$
   An important special case is for classical distributions over bits, where the Pinsker inequality implies
   $$D\left(\begin{pmatrix} p+\epsilon \\ 1-p-\epsilon \end{pmatrix} \middle\| \begin{pmatrix} p \\ 1-p \end{pmatrix}\right) \geq \frac{2}{\ln 2}\epsilon^2. \tag{3}$$
   A related inequality is the Chernoff bound, which is a way of showing that sums of many independent random variables are exponentially unlikely to be far from their mean. One version of this bound states that if $X_1, \ldots, X_n$ are i.i.d. random variables such that $\Pr[X_i = 1] = p$ and $\Pr[X_i = 0] = 1 - p$, then
   $$\Pr\left(\frac{1}{n}\sum_{i=1}^{n} X_i \geq p + \epsilon\right) \leq e^{-2n\epsilon^2}. \tag{4}$$

   Derivations of eq. (3) and eq. (4) (not needed for the rest of the problem) can be found on wikipedia, and you may take these equations as given.

   On an earlier pset, you showed how a Chernoff-like bound could be derived from Pinsker's inequality using the method of types. Here you will *assume* the Chernoff bound and use this to prove Pinsker's inequality.

   (a) Prove eq. (2). There are two possible routes. One is to use eq. (4) and the quantum Stein's Lemma. Another is to use the monotonicity of relative entropy and eq. (3). Pick one of these, or come up with another.

   (b) The Pinsker inequality can be used to derive approximate versions of various entropic conditions. Prove the following:

      i. If $S(\rho) \leq \epsilon$ then ρ is close in trace distance to a pure state, where "close" means the distance goes to 0 as $\epsilon \to 0$. [Hint: let $\rho = \sum_i \lambda_i \psi_i$ for $\lambda_1 \geq \lambda_2 \geq \cdots$ and show $D(\psi_1\|\rho) \leq S(\rho)$.]

      ii. If $I(A;B)_\rho \leq \epsilon$ then $\rho_{AB} \approx \rho_A \otimes \rho_B$ where again $\approx$ means close in trace distance. [Hint: Pinsker!]

      iii. For this last part, there is nothing to turn in. If $|H(A|B)| \leq \epsilon$ then there is no simple structural statement we can make (in the quantum case). Think about why this is true. We will later see that $I(A;B|C) \leq \epsilon$ implies a structural property about quantum states but this is very far from obvious.