Q. Inf. Science 3 (8.372) — Fall 2024

Assignment 9

Due: **Tuesday**, *Nov 19, 2024 at* ***9pm***

Turning in your solutions: Upload a single pdf file to gradescope.

1. **Random states**

   (a) Consider a collection of unit vectors $|v_1\rangle, \ldots, |v_n\rangle \in \mathbb{C}^d$. Everyone knows that if the vectors are orthogonal then we must have $n \le d$. Suppose instead we have $\langle v_i|v_j\rangle = \alpha$ for each $i \ne j$ where α is a fixed real number in the interval $(0,1)$. Show that we again must have $n \le d$. As a hint, the Gram matrix has entries $G_{i,j} = \langle v_i|v_j\rangle$; consider its rank.

   (b) Choose unit vectors $|v\rangle, |w\rangle \in \mathbb{C}^d$ uniformly at random. Show that

   $$\Pr\left[|\langle v|w\rangle|^2 \ge \frac{1}{2}\right] \le c^{-d} \tag{1}$$

   for some constant $c > 1$. As a hint, consider higher moments of $|\langle v|w\rangle|^2$. (Note that the choice of 1/2 here was arbitrary, and the same result holds for any number between 0 and 1.)

   (c) Show that one can construct unit vectors $|v_1\rangle, \ldots, |v_n\rangle \in \mathbb{C}^d$ with $n \ge \exp(\Omega(d))$ and $|\langle v_i|v_j\rangle|^2 \le 1/2$ for each $i \ne j$.

   (d) Explain how to use the construction in the previous part for *equality testing*. In this problem, Alice and Bob each receive inputs $x, y \in [n]$, and each send $\log d$ qubits to Charlie, who has to guess whether $x = y$ or $x \ne y$. If $x = y$ then Charlie should output EQUAL with probability c, while if $x \ne y$ Charlie should output EQUAL with probability s, and we should have $c - s \ge \Omega(1)$.

   (e) *[Optional and way too hard for the pset.]* Construct rank-$d/2$ dimensional projectors $\Pi_1, \ldots, \Pi_n$ in $\mathbb{C}^d$ with $\operatorname{tr} \Pi_i \Pi_j \le \frac{2}{3}d$ and $n \ge \exp(\Omega(d^2))$.
   [Hint: If you want a spoiler, look at Lemma III.5 of quant-ph/0407049.]

2. **Spectrum estimation**

   In Schur-Weyl duality, we decompose

   $$\mathbb{C}^d \cong \bigoplus_{\lambda \in \mathrm{Par}(n,d)} \mathcal{Q}_\lambda \otimes \mathcal{P}_\lambda. \tag{2}$$

   Let the corresponding projectors be Π_λ, and let the unitary performing this change of basis be U_{SW}. It turns out that

   $$U_{\mathrm{SW}} \rho^{\otimes n} U_{\mathrm{SW}}^\dagger = \sum_{\lambda \in \mathrm{Par}(n,d)} |\lambda\rangle\!\langle\lambda| \otimes q_\lambda(\rho) \otimes I_{\mathcal{P}_\lambda} \tag{3}$$

so that

$$\operatorname{tr}\left[\Pi_\lambda \rho^{\otimes n}\right] = \operatorname{tr}[q_\lambda(\rho)] \dim \mathcal{P}_\lambda \tag{4}$$

Here q_λ is the representation matrix for the group $U(d)$ acting on the irrep $\mathcal{Q}_\lambda$. We can extend it to act on non-unitary matrices (like ρ) by treating it as a polynomial in the entries of ρ. You will not need to carry this out but can rely on the characterization of q_λ below in (10).

Let the sorted eigenvalues of ρ be denoted $r = (r_1, \dots, r_d)$. So $r_1 \geq r_2 \geq \cdots r_d \geq 0$. Let $\bar{\lambda} := \lambda/n$. In this problem we will prove that

$$\operatorname{tr}\left[\Pi_\lambda \rho^{\otimes n}\right] \leq n^{O(d^2)} \exp\left(-nD(\bar{\lambda}\|r)\right), \tag{5}$$

implying that measuring Π_λ is an effective way of estimating the spectrum of ρ.

(a) Prove that

$$\dim \mathcal{P}_\lambda \leq \binom{n}{\lambda} := \frac{n!}{\lambda_1! \cdots \lambda_d!}. \tag{6}$$

Using problem 2 from pset 2, relate this to $\exp\left(nH(\bar{\lambda})\right)$.

As a hint, the basis elements of $\mathcal{P}_\lambda$ are labeled by the Standard Young Tableaux (SYT) meaning ways of filling λ with the numbers $1, \dots, n$ with the numbers strictly increasing from left to right and from top to bottom. For example, the SYT of shape $\lambda = (4, 2)$ are

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline 5&6\\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1&2&3&5\\ \hline 4&6\\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1&2&3&6\\ \hline 4&5\\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1&2&4&5\\ \hline 3&6\\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1&2&4&6\\ \hline 3&5\\ \cline{1-2}\end{array} \tag{7}$$

$$\begin{array}{|c|c|c|c|}\hline 1&2&5&6\\ \hline 3&4\\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1&3&4&5\\ \hline 2&6\\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1&3&4&6\\ \hline 2&5\\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1&3&5&6\\ \hline 2&4\\ \cline{1-2}\end{array} \tag{8}$$

(b) In this part you will prove that

$$\operatorname{tr}[q_\lambda(\rho)] \leq n^{O(d^2)} r^\lambda \quad \text{where} \quad r^\lambda := r_1^{\lambda_1} \cdots r_d^{\lambda_d} \tag{9}$$

Here you will need to use the fact that

$$\operatorname{tr}[q_\lambda(\rho)] = \sum_{\mu \text{ a SSYT of shape } \lambda} r^{w(\mu)} \tag{10}$$

A SSYT (semistandard Young tableau) of shape λ is a way of filling λ with the numbers $1, \dots, d$ such that the numbers are strictly increasing from top to bottom and weakly increasing (i.e. nondecreasing) from left to right. So the SSYT of shape $\lambda = (2, 1)$ with $d = 3$ are

$$\begin{array}{|c|c|}\hline 1&1\\ \hline 2\\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\ \hline 2\\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&3\\ \hline 2\\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\ \hline 3\\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&3\\ \hline 3\\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2&2\\ \hline 3\\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2&3\\ \hline 3\\ \cline{1-1}\end{array}, \tag{11}$$

The "weight" of μ is denoted $w(\mu)$ and is the unnormalized type, meaning the vector of frequencies of each letter. So $w\left(\begin{array}{|c|c|}\hline 1&1\\ \hline 2\\ \cline{1-1}\end{array}\right) = (2, 1, 0)$ (assuming $d = 3$) and

$$\operatorname{tr}\left[q_{(2,1)}(\rho)\right] = r_1^2 r_2 + r_1 r_2^2 + 2r_1 r_2 r_3 + r_1 r_3^2 + r_2^2 r_3 + r_2 r_3^2 \tag{12}$$

i. Prove that the number of SSYT of shape λ are $\leq n^{O(d^2)}$. As a hint, just consider one row at a time.

ii. Let w be the weight of a SSYT μ of shape λ. Prove that $w_1 + \cdots + w_k \leq \lambda_1 + \cdots + \lambda_k$ for each $k = 1, \ldots, d$.
When this happens we can also write $w \preceq \lambda$ and say that w is "majorized by" λ. As a hint, what is the lowest row that the number k can appear in?

iii. Prove (9).