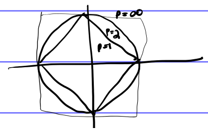
Metrics on states and prob. distributions

Norms $\| \cdot \|$

satisfying
1) $\|cv\| = |c| \cdot \|v\|$ for $c \in \mathbb{C}$
2) $\|v + w\| \le \|v\| + \|w\|$
3) $\|v\| = 0 \Leftrightarrow v = 0$

$\|v\|^2 = \langle v, v\rangle$? only for 2-norm

L_p $\quad x \in \mathbb{C}^d \quad \|x\|_{L_p} = \|x\|_p = \left(\sum_{i=1}^d |x_i|^p\right)^{1/p}$



$L_1 \quad \sum_i |x_i|$

$L_2 \quad$ Euclidean

$L_\infty \quad \max_i |x_i|$

S_p = Schatten-p $\quad \|X\|_{S_p} = \|X\|_p = \|\,\mathrm{svals}(X)\,\|_{L_p}$

$\|X\|_{S_1} = \mathrm{tr}\sqrt{X^\dagger X} = \mathrm{tr}\,|X|$ if X is normal

$\|X\|_{S_\infty}$ = op. norm = biggest sval = $\|X\|$

unit sphere / ball $\quad S(\ ),\ B(\ )$

pure state QM $\quad S(L_2)$

probability dists $\quad S(L_1) \cap \mathbb{R}^d_{\geq}$

density matrices $\quad S(S_1) \cap$ psd

measurement ops $\quad B(S_\infty)$

L_p and L_q dual if $\frac{1}{p} + \frac{1}{q} = 1$

S_p & S_q " " "

e.g. $1, \infty$

$2, 2$

Comparing prob dists

total variation distance

$T(p,q) = \frac{1}{2}\|p-q\|_1 = \max_{\|x\|_\infty \le 1} \frac{1}{2}\langle p-q, x\rangle = \max_{S \subseteq [d]} |p(S) - q(S)|$

fidelity

$\langle \sqrt{p}, \sqrt{q}\rangle = \sum_x \sqrt{p(x) q(x)} =: F(p,q)$

$1 - F \le T \le \sqrt{2(1-F)}$

$F(p_1 \otimes p_2, q_1 \otimes q_2) = F(p_1, q_1)\, F(p_2, q_2)$

$\Rightarrow 1 - T(p^{\otimes n}, q^{\otimes n}) \sim e^{-cn}$

7. States

pure states $\| |\alpha\rangle - |\beta\rangle \|_2 = \sqrt{2(1-\mathrm{Re}\langle\alpha|\beta\rangle)}$

not great. $F(\alpha,\beta) = |\langle\alpha|\beta\rangle|$ avoids phase dependence

mixed states

$$T(\rho,\sigma) = \frac{1}{2}\|\rho-\sigma\|_1 = \max_{0\le M\le I} \mathrm{tr}\, M(\rho-\sigma)$$

properties $T(V\rho V^\dagger, V\sigma V^\dagger) = T(\rho,\sigma)$

$T(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \le T(\rho,\sigma)$ pset

note that $\mathcal{E}(\rho) = \mathrm{tr}(M\rho)\,|0\rangle\langle 0| + \mathrm{tr}((I-M)\rho)\,|1\rangle\langle 1|$
saturates this

fidelity of mixed states

$$F(\rho,\sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_{S_1} = \mathrm{tr}\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}$$

pset $1-F \le T \le \sqrt{1-F^2}$ $0\le F\le 1$

and $F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \ge F(\rho,\sigma)$ arccos F is a metric

Uhlmann's thm

$F(\rho,\sigma) = \max |\langle\alpha|\beta\rangle|$ over $|\alpha\rangle_{AB}, |\beta\rangle_{AB}$ s.t. $\alpha_A=\rho$, $\beta_B=\sigma$

by purification uniqueness, we can assume $|\alpha\rangle$ is fixed

$|\phi^\rho\rangle = (\sqrt{\rho}\otimes I)|\Gamma\rangle$ $\quad |\Gamma\rangle = \sum_i |i\rangle\otimes|i\rangle$

Pf

$|\alpha\rangle = |\phi^\rho\rangle \qquad |\beta\rangle = (I\otimes U)|\phi^\sigma\rangle$

$$\text{RHS} = \max_U |\langle\phi^\rho|\, I\otimes U\, |\phi^\sigma\rangle|$$

$$= \max_U |\langle\Gamma| \underbrace{(\sqrt{\rho}\otimes I)(I\otimes U)(\sqrt{\sigma}\otimes I)}_{\sqrt{\rho}\sqrt{\sigma}\otimes U} |\Gamma\rangle|$$

$$= \max_U |\mathrm{tr}\sqrt{\rho}\sqrt{\sigma}U^T| \underset{*}{=} \|\sqrt{\rho}\sqrt{\sigma}\|_1 = F$$

Justifying $\cancel{\#}$ S_1/S_∞ duality

$A = VDW^\dagger$ $\quad \max_U |\mathrm{tr}\, AU| = |\mathrm{tr}\, D W^\dagger U V|$

$= \sum_i D_{ii} \underbrace{(W^\dagger U V)_{ii}}_{\leq 1} \leq \|A\|_1$

achieved if $W^\dagger U V = I$

No-go for noisy bit commitment

After commit states are $|\psi_0\rangle_{AB}$ or $|\psi_1\rangle_{AB}$

$T(\psi_0^B, \psi_1^B) \leq \epsilon$

$\Rightarrow F(\psi_0^B, \psi_1^B) \geq 1-\epsilon$

$\Rightarrow \exists\, U$ s.t. $F(\underbrace{(U\otimes I)|\psi_0\rangle}_{|\psi_{fake}\rangle}, |\psi_1\rangle) \geq 1-\epsilon$

$T(\psi_{fake}, \psi_1) \leq \sqrt{2\epsilon}$

$T(\mathrm{reveal}(\psi_{fake}), \mathrm{reveal}(\psi_1)) \leq \sqrt{2\epsilon}$

What do T & F measure?

$\begin{pmatrix}1\\0\end{pmatrix}$ vs $\begin{pmatrix}1-\epsilon\\ \epsilon\end{pmatrix}$ or $\begin{pmatrix}1/2\\1/2\end{pmatrix}$ vs $\begin{pmatrix}1/2+\epsilon\\1/2-\epsilon\end{pmatrix}$

or $|0\rangle$ vs $\cos\theta|0\rangle + \sin\theta|1\rangle$