Classical compression

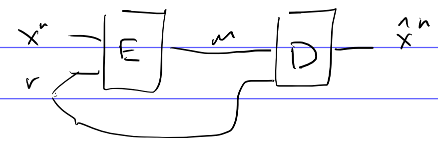
$H(p) = H(X)_p = -\sum_x p(x)\log p(x)$

$T^n_{p,\delta} = \{ x^n = (x_1,\dots,x_n) : \left|-\frac{1}{n}\log p^{\otimes n}(x^n) - H(x)\right| \le \delta \}$

$\epsilon = 1 - p^{\otimes n}(T^n_{p,\delta}) \lesssim n^{O(1)}\, 2^{-n\delta} \to 0$ as $n \to \infty$

converse

- compress to k bits
- derandomize



pick r to maximize $\Pr[\hat{x}^n = x^n \mid r]$

- $S \subseteq \Sigma^n$ = set decoded correctly

$|S| \le 2^k$

$p^n(S) \le p^n(T^n_{p,\delta} \cap S) + p^n(\overline{T}^n_{p,\delta})$

$\le 2^k\, 2^{-nH(p)+n\delta} + \epsilon$

$\to 0$ if $\frac{k}{n} < H(p)$

Q compression

$S(\rho) = H(\mathrm{eig}(\rho)) = -\mathrm{tr}[\rho\log\rho]$

$0 \le S(\rho) \le \log d$

$S(\rho) = 0 \iff \mathrm{eig}(\rho) = (1,0,\dots,0) \iff \rho = |\psi\rangle\langle\psi|$

$S(\rho) = \log d \iff \rho = I/d$

$S(X) = \rho_X$

$S(X|Y) = S(XY) - S(Y)$ (this defn carries over)

can be negative

$I(X:Y) = S(X) + S(Y) - S(XY) = S(X) - S(X|Y)$

typical subspaces & projectors

$$\rho = \sum_{x=1}^{d} \lambda_x \, |v_x\rangle\langle v_x| \qquad \rho^{\otimes n} = \sum_{x^n} \lambda_{x_1} \cdots \lambda_{x_n} \, |v_{x^n}\rangle\langle v_{x^n}|$$

$$\Pi^n_{\rho,\delta} = \sum_{x^n \in T^n_{\lambda,\delta}} |v_{x^n}\rangle\langle v_{x^n}| \qquad |v_{x^n}\rangle = |v_{x_1}\rangle \otimes \cdots \otimes |v_{x_n}\rangle$$

typ. proj. $\quad$ supp $\Pi^n_{\rho,\delta}$ is typ. subspace

$$\mathrm{tr}\, \rho^{\otimes n} \Pi^n_{\rho,\delta} = \sum_{x^n} \lambda_{x_1} \cdots \lambda_{x_n} \, 1_{x^n \in T^n_{\rho,\delta}} = \lambda^n (T^n_{\lambda,\delta})$$

q compression?

1) $\rho^{\otimes n} - \boxed{\mathcal{E}} - \boxed{\mathcal{D}} - \sigma \approx \rho^{\otimes n}$

2) $|v_{x^n}\rangle - \boxed{\mathcal{E}} - \boxed{\mathcal{D}} - \sigma \qquad \mathbb{E}_{x^n \sim \lambda^n} F(|v_{x^n}\rangle\langle v_{x^n}|, \sigma) \approx 1$

3) $\rho = \sum_i p_i |\omega_i\rangle\langle\omega_i| \qquad |\omega_{i^n}\rangle - \boxed{\mathcal{E}} - \boxed{\mathcal{D}} - \sigma \qquad \mathbb{E}_{i^n \sim p^n} F(|\omega_{i^n}\rangle, \sigma) \approx 1$

4) $|\phi_\rho^{AR}\rangle^{\otimes n}$ — R / ; — / $\boxed{\mathcal{E}}$ — $\boxed{\mathcal{D}}$ — $\}\ \sigma \qquad F(|\phi_\rho\rangle^{\otimes n}, \sigma) \approx 1$

— / — means $\equiv$, i.e. n qubits

① is too weak $\qquad$ 2–4 are all good, ≈ equiv

Schumacher – Jozsa compression

$$\{\Pi^n_{\rho,\delta},\ I - \Pi^n_{\rho,\delta}\} \qquad \mathrm{tr}\, \Pi^n_{\rho,\delta} \le \exp(n(S(\rho)+\delta))$$

alg. coding $\quad \rho = \dfrac{I + \epsilon Z}{2}$

$$F\Big(\sum_i p_i |i\rangle\langle i| \otimes \alpha_i,\ \sum_i p_i |i\rangle\langle i| \otimes \beta_i\Big) =$$

$$\Big\| \Big(\sum_i \sqrt{p_i}\, |i\rangle\langle i| \otimes \sqrt{\alpha_i}\Big)\Big(\sum_j \sqrt{p_j}\, |j\rangle\langle j| \otimes \sqrt{\beta_j}\Big) \Big\|_1$$

$$= \sum_i p_i \|\sqrt{\alpha_i}\sqrt{\beta_i}\|_1$$

④ → ③ → ②

4→3 measure R and get ensemble.

3→2 pick eigen basis

$$F\Big(\sum_{x^n} |x^n\rangle\langle x^n| \otimes \lambda_{x^n} |v_{x^n}\rangle\langle v_{x^n}|,$$

$$\sum_{x^n} |x^n\rangle\langle x^n| \otimes \lambda_{x^n}\, \mathcal{D}(\mathcal{E}(|v_{x^n}\rangle\langle v_{x^n}|))\Big) = \sum_{x^n} \lambda_{x^n} F(|v_{x^n}\rangle, \mathcal{D}(\mathcal{E}(|v_{x^n}\rangle)))$$

$$\sum_{x^n} \lambda_{x^n} \langle v_{x^n} | \mathcal{D}(\mathcal{E}(|v_{x^n}\rangle\langle v_{x^n}|)) | v_{x^n}\rangle$$