Pf sketch of Chernoff-Stein thm

LRT (likelihood ratio test)

$X^n \sim p^n$ or q^n $\qquad W(x^n) = \log \frac{p^n(x^n)}{q^n(x^n)}$

$\underset{x^n\sim p^n}{\mathbb{E}} W = n\, D(p\|q) \qquad \underset{x^n\sim q^n}{\mathbb{E}} W = -n\, D(q\|p)$

$A = \{x^n : W \geq n(D(p\|q) - \delta)\} \qquad x^n \in A \Leftrightarrow q^n(x^n) \leq e^{-n(D-\delta)} p^n(x^n)$

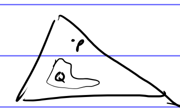
$q^n(A) \leq e^{-n(D-\delta)} p^n(A) \leq e^{-n(D-\delta)}$

$p^n(A) \to 1$ by law of large numbers

multiple hypothesis testing

$x^n \sim p^n$ or q^n for $q \in Q$

rate $= \min_{q\in Q} D(p\|q)$



quantum relative entropy

$D(\rho\|\sigma) = \mathrm{tr}[\rho(\log\rho - \log\sigma)]$

$\beta_\epsilon^n = \min\{\mathrm{tr}\, M\sigma^{\otimes n} : \mathrm{tr}\, M\rho^{\otimes n} \geq 1-\epsilon\}$

q Stein $\qquad \lim_{n\to\infty} -\frac{1}{n}\log \beta_\epsilon^n = D(\rho\|\sigma)$

$D(\rho\|\sigma) = \infty \Leftrightarrow \mathrm{supp}\,\rho \not\subseteq \mathrm{supp}\,\sigma$

e.g. $\rho = |\psi\rangle\langle\psi| \qquad \sigma = |\phi\rangle\langle\phi|$

$A = |\psi^\perp\rangle\langle\psi^\perp| \qquad B = I - A \qquad M = A \otimes B^{\otimes n-1} + B \otimes A \otimes B^{\otimes n-2} + \cdots + B^{\otimes n-1} \otimes A$

Application to entropies

$D(\rho\|\sigma) \geq 0 \qquad S(A) \leq \log d$

$I(A;B) \geq 0 \qquad S(A) \geq S(A|B)$

Pf of q Stein

want M s.t. $\operatorname{tr} \rho^{\otimes n} M \geq \alpha$

$\operatorname{tr} \sigma^{\otimes n} M \lesssim e^{-nR}$ $\qquad R \approx D(\rho \| \sigma)$

optimal $M = [\alpha^{-1} \rho^{\otimes n} - 2^{nR} \sigma^{\otimes n} \geq 0]$

following Bjelakovic et al.

$\rho = \sum_x r_x |\alpha_x\rangle\langle\alpha_x|$ $\qquad \sigma = \sum_x s_x |\beta_x\rangle\langle\beta_x|$

$\Pi^n_{\rho\|\sigma,\delta} = \sum_{x^n : \left|\frac{1}{n}\sum_{i=1}^n \log s_{x_i} - \operatorname{tr} \rho \log \sigma\right| \leq \delta} \beta_{x^n}$ $\qquad \beta_{x^n} = \beta_{x_1} \otimes \cdots \otimes \beta_{x_n}$

cf. $\Pi^n_{\rho,\delta} = \sum_{x^n \text{ s.t. } \left|\frac{1}{n}\sum_{i=1}^n -\log r_{x_i} - S(\rho)\right| \leq \delta} |\alpha_{x^n}\rangle\langle\alpha_{x^n}|$

$\operatorname{tr} \rho^{\otimes n} \Pi_{\rho\|\sigma} \geq 1 - \epsilon$ $\qquad$ law of large numbers

$[\Pi, \sigma^{\otimes n}] = 0$

$2^{n(\operatorname{tr} \rho \log \sigma - \delta)} \Pi \leq \Pi \sigma^{\otimes n} \Pi \leq 2^{n(\operatorname{tr} \rho \log \sigma + \delta)} \Pi$

Achievability: $\quad M = \Pi^n_{\rho\|\sigma,\delta} \Pi^n_{\rho,\delta} \Pi^n_{\rho\|\sigma,\delta}$

$\operatorname{tr} \rho^{\otimes n} M \to 1$ using gentle measurement lemma

$\operatorname{tr} \sigma^{\otimes n} M \leq \operatorname{tr} \Pi^n_{\rho,\delta} 2^{n(\operatorname{tr} \rho \log \sigma + \delta)} \Pi^n_{\rho\|\sigma,\delta} \leq 2^{n(S(\rho)+\delta)} 2^{n(\operatorname{tr} \rho \log \sigma + \delta)} = \exp(-n(D(\rho\|\sigma) - 2\delta))$

Converse

suppose $\operatorname{tr} \rho^{\otimes n} M \geq \alpha$

$\sigma^{\otimes n} \geq \Pi_{\rho\|\sigma} 2^{n(\operatorname{tr} \rho \log \sigma - \delta)}$ $\qquad \rho^{\otimes n} \Pi^n_{\rho,\delta} \leq 2^{-n(S(\rho) - \delta)} \Pi^n_{\rho,\delta}$

$\operatorname{tr} M \sigma^{\otimes n} \geq \operatorname{tr}(M \Pi_{\rho\|\sigma}) 2^{n(\operatorname{tr} \rho \log \sigma - \delta)}$

$\operatorname{tr} M \Pi_{\rho\|\sigma} = \operatorname{tr} \Pi_{\rho\|\sigma} M \Pi_{\rho\|\sigma}$

$\geq \operatorname{tr} \Pi_{\rho\|\sigma} M \Pi_{\rho\|\sigma} \Pi_\rho$

$\geq \underbrace{\operatorname{tr} \Pi_{\rho\|\sigma} M \Pi_{\rho\|\sigma} \Pi_\rho \rho^{\otimes n}} \, 2^{-n(S(\rho) - \delta)}$ $\qquad \rho^{\otimes n} = \overbrace{\Pi_\rho \rho^{\otimes n}}^{A} + \overbrace{(I - \Pi_\rho) \rho^{\otimes n}}^{B}$

$\operatorname{tr} M \left(\Pi_{\rho\|\sigma} \left(\rho^{\otimes n} - \underbrace{(I - \Pi_\rho) \rho^{\otimes n}}_{\operatorname{tr} \leq \epsilon}\right) \Pi_{\rho\|\sigma}\right)$

$\geq (\alpha - 2\sqrt{\epsilon} - \epsilon)$

$\operatorname{tr} M \sigma^{\otimes n} \geq (\alpha - 2\sqrt{\epsilon} - \epsilon) \, 2^{-n(D(\rho\|\sigma) + 2\delta)}$

Cor $\quad D(\rho\|\sigma) \geq D(\mathcal{E}(\rho) \| \mathcal{E}(\sigma))$

Cor strong subadditivity

$I(A:C|B) = I(A:BC) - I(A:B) = D(\rho_{ABC} \| \rho_A \otimes \rho_{BC}) - D(\rho_{AB} \| \rho_A \otimes \rho_B) \geq 0$

monotonicity under tr_C