# Noisy channel coding

$X - \boxed{N} - Y \qquad p(x,y) = \pi(x)\, N(y|x)$

Shannon $\quad C(N) = \max_{\pi} I(X;Y)_p$

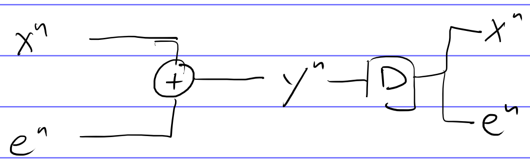
$$= \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \left( \begin{array}{l} \#\text{ messages that can be sent} \\ \text{through } N^n \text{ with error} \leq \epsilon \end{array} \right)$$

e.g. BSC

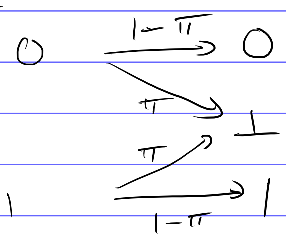
$y = x + e \mod 2$

$\Pr[e=0] = 1-\pi \qquad \Pr[e=1] = \pi$

$I(X;Y) = H(Y) - H(Y|X) = 1 - H_2(\pi)$




erasure




$H(Y|X) = H_2(\pi)$

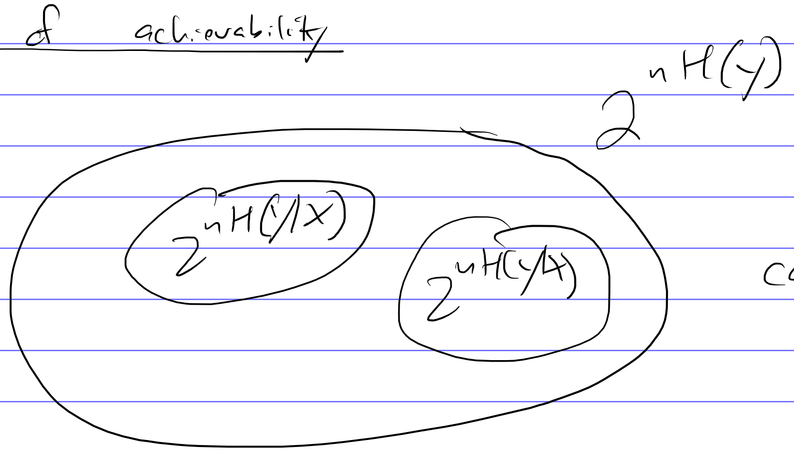
$H(Y) = 1-\pi + H_2(\pi)$

$I(X;Y) = 1 - \pi$

just as good as having a feedback channel

Gaussian $\quad x \in \mathbb{R} \qquad e \sim N(0, \sigma^2) \qquad y = x + e$

Pf of achievability




can fit $\approx 2^{nI(X;Y)}$

code words

$q_x(y) := N(y|x) \qquad q(y) = \sum_x \pi(x) q_x(y)$

$$D(q_x \| q) = -H(q_x) - \langle q_x, \log q \rangle$$

$$\sum_x \pi(x) D(q_x \| q) = -\sum_x \pi(x) H(q_x) - \langle q, \log q \rangle$$

$$= H(Y) - H(Y|X) = I(X;Y)$$

consider codewords $x^n(m) \in T_\pi$ (type $\pi$)

$m = 1, \ldots, M$ (later $M \approx 2$

$$N^n(x^n(m)) = q_{x_1} \otimes q_{x_2} \otimes \cdots \otimes q_{x_n}$$

up to permutation $q_1^{\otimes n\pi_1} \otimes \cdots \otimes q_d^{\otimes n\pi_d}$

so $D\left( N^n(x^n(m)) \,\|\, q^{\otimes n} \right) = n\, I(X;Y)$

Stein's lemma $\Rightarrow \exists$ test $A(m)$

s.t. $N^n(x^n(m))(A(m)) \geq 1 - \epsilon$

$q^{\otimes n}(A(m)) \leq 2^{-n(I(X;Y) - \delta)}$

choose $x^n(1), x^n(2), \ldots, x^n(M)$ iid from $\pi^n$ or from $T_\pi$

Alice encodes $m \to x^n(m)$

Bob tests $A(1), A(2), \ldots, A(M)$

$\mathbb{E}_{\text{codebooks}} \; \mathbb{E}_{m \in [M]} \; N^n(x^n(m)) \approx q^{\otimes n}$

so $\Pr[\text{wrong test accepting}] \leq M \cdot 2^{-n(I(X;Y) - \delta)}$

$\Pr[\text{rejecting true message}] \leq \epsilon$

## Quantum version

$N$ a CQ channel $x \to \rho_x$

$C(N) = \max_p I(X;Q)_\omega \qquad \omega^{XQ} = \sum_x p(x) |x\rangle\langle x|^X \otimes \rho_x^Q$

e.g. $\nwarrow\nearrow\downarrow$ states have capacity 1

Pf needs non-commutative union bound

$\omega \geq 0 \quad P_1, \ldots, P_L \quad \operatorname{tr}[\omega] - \operatorname{tr}[P_L \cdots P_1 \omega P_1 \cdots P_L] \leq 2\sqrt{\sum_i \operatorname{tr}[(I - P_i)\omega]}$

$\operatorname{tr}\omega \leq 1$ projectors