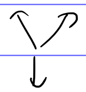
Classical messages over q. channels

$\mathcal{N}$ a CQ channel $\quad x \to \rho_x$

$$C(\mathcal{N}) = \max_p I(X;Q)_\omega \qquad \omega^{XQ} = \sum_x p(x)|x\rangle\langle x|^X \otimes \rho_x^Q$$

$$\rho = \omega^Q = \sum p(x)\rho_x$$

e.g.  states have capacity 1

Pf needs non-commutative union bound

$\omega \geq 0$ $\quad P_1, \ldots, P_L$ $\quad \mathrm{tr}[\omega] - \mathrm{tr}[P_L \cdots P_1 \omega P_1 \cdots P_L] \leq 2\sqrt{\sum_i \mathrm{tr}[(I - P_i)\omega]}$

$\mathrm{tr}\,\omega \leq 1$ $\quad$ projectors

Let p achieve max in $C(\mathcal{N})$ expression

sample $\quad x^n(1), \ldots, x^n(M)$ from p^n

$\sigma_m := \rho_{x_1(m)} \otimes \cdots \otimes \rho_{x_n(m)}$

Π_m = conditionally typical projector for σ_m

i.e. if $t = \text{type}(x^n(m))$ then $\Pi_m = \pi\left[\bigotimes_{i=1}^d \Pi^{nt_i}_{\rho_i,\delta}\right]\pi^{-1}$ for some permutation π

$\mathrm{tr}\,\sigma_m \Pi_m \geq 1 - \epsilon$

$\mathop{\mathbb{E}}_{x^n(m)} \sigma_m = \rho^{\otimes n}$

$$\sum_{m' \neq m} \mathop{\mathbb{E}}_{x^n(m)} \mathrm{tr}\,\Pi_{m'}\sigma_m \leq M\,\mathrm{tr}\,\rho^{\otimes n}\Pi_{m'} \leq M \exp\left(-\sum_i n t_i D(\rho_i \| \rho)\right)$$

$$\lesssim \exp(nR - n I(X;Q)_\omega - n\delta)$$

using $t \approx \pi$

specifically $\quad n\sum_i t_i D(\rho_i\|\rho) - I(X,Q) = n\sum_i (t_i - \pi_i) D(\rho_i \| \rho)$

$$\leq n \|t - \pi\|_1 \max_i D(\rho_i\|\rho)$$

$$\leq n\,\delta \log d$$

Converse

$M - X^n - Y^n - \hat{M}$

$\Pr[M \neq \hat{M}] \leq \epsilon \qquad M \in \{0,1\}^{nR}$

Lemma: Fano's inequality $\qquad H(M|\hat{M}) \leq \epsilon nR + 1$

Pf $\quad d = 2^{nR}$

let $p = \Pr[\cdot \mid \hat{M}] \qquad p = (1-\delta)\,1_{\hat{M}} + \delta q \qquad \delta \leq \epsilon$

$H(p) = -(1-\delta)\log(1-\delta) - \sum_x \delta q(x) \log\delta + \log q(x) \qquad q(\hat{M}) = 0$

$= H_2(\delta) + \delta H(q)$

$\leq 1 + \epsilon \log d$

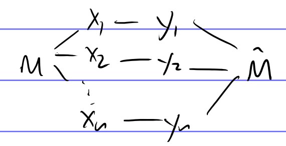
More useful is Fannes' ineq: $\quad |H(p) - H(q)| \leq H_2(\epsilon) + \epsilon \log d \qquad \epsilon = \frac{1}{2}\|p - q\|_1$

$|S(\rho) - S(\sigma)| \leq H_2(\epsilon) + \epsilon \log d$

random $M \to H(M) = nR$

$I(M;\hat{M}) = H(M) - H(\hat{M}) \geq nR - \epsilon nR - 1$

need to compare $I(M;\hat{M})$ with $I(X^n;Y^n)$



Def CMI - conditional mutual information

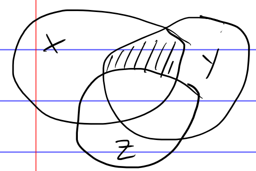
$I(X;Y|Z) = \sum_z p_Z(z)\, I(X;Y|Z=z) \geq 0$

$= H(X|Z) + H(Y|Z) - H(XY|Z)$

$= H(XZ) - H(Z) + H(YZ) - H(Z) - H(XYZ) + H(Z)$

$= H(XZ) + H(YZ) - H(XYZ) - H(Z)$

$= I(X;YZ) - I(X;Z)$



chain rule for mutual information

$I(X;YZ) = I(X;Z) + I(X;Y|Z)$

$I(X;Y_1 \ldots Y_n) = I(X;Y_1) + I(X;Y_2|Y_1) + \ldots + I(X;Y_n|Y_1 \ldots Y_{n-1})$

$\text{CMI} = 0 \iff X - Z - Y$ is a Markov chain

$\iff p(x,y,z) = p(z)\,p(y|z)\,p(x|z)$

quantumly $I(A;B|C) = S(AC) + S(BC) - S(ABC) - S(C) \geq 0$ SSA

q Markov state $\iff$ CMI=0 characterization later — Strong subadditivity

Pf via monotonicity of relative entropy

data processing inequality: $X - Y - Z \Rightarrow I(X;Z) \geq I(X;Y)$

Pf $I(X;Z) = I(X;YZ) - I(X;Y|Z)$

$I(X;Y) = I(X;YZ) - I(X;Z|Y)$

$I(X;Z) - I(X;Y) = I(X;Z|Y) - I(X;Y|Z)$

$= I(X;Z|Y) \geq 0$ ∎

works quantumly too.

$(1-\epsilon)nR - 1 \leq I(M;\hat{M}) \leq I(X^n;Y^n) \overset{*}{\leq} \sum_{j=1}^{n} I(X_j;Y_j) \leq nC \Rightarrow R \leq \frac{C}{1-\epsilon}$

(*) $I(X^n;Y^n) = H(Y^n) - H(Y^n|X^n)$

$H(Y^n|X^n) = \sum_{j=1}^{n} H(Y_j|X^n Y_1 \ldots Y_{j-1})$

$= \sum_j H(Y_j|X_j)$

$H(Y^n) \leq \sum_j H(Y_j)$

All works for CQ channels as well

$I(X^n;Q^n) = S(Q^n) - S(Q^n|X^n)$

$\leq \sum_j S(Q_j) - \sum_j S(Q_j|X_j) \leq n\chi$