State tomography

$\rho \in D_d$ ($d\times d$ density matrices)

Suppose we are given $\rho^{\otimes n}$ and want to guess $\hat\rho$ s.t. $\|\rho-\hat\rho\|_1 \le \epsilon$

$d=2$ measure σ_i $i=1,2,3$ on $\frac{n}{3}$ copies each

learn $\operatorname{tr}\rho\sigma_i \pm O(\frac{1}{\sqrt n})$ $\epsilon \sim 1/\sqrt{n}$ or $n \gtrsim \frac{1}{\epsilon^2}$

for general d, ρ has d^2-1 real parameters

guess $n \gtrsim d^2$

In fact $n \sim \frac{d^2}{\epsilon^2}$ necessary and sufficient

Sufficient — we'll see later

for now, note a pitfall

If $\sigma_1,\dots,\sigma_{d^2-1}$ satisfy $\operatorname{tr}\sigma_i\sigma_j = d\,\delta_{ij}$ then $\rho = \sum_i \alpha_i\sigma_i$

$\operatorname{tr}\rho\sigma_i = d\alpha_i$

measure $\frac{n}{d^2}$ copies $\Rightarrow$ learn $d\alpha_i \pm \frac{d}{\sqrt n}$ $|\hat\alpha_i - \alpha_i| \sim 1/\sqrt{n}$

$\hat\rho = \sum_i \hat\alpha_i\sigma_i$ $\rho-\hat\rho = \sum_i(\alpha_i-\hat\alpha_i)\sigma_i$ $\|\rho-\hat\rho\|_2^2 = \sum_i(\alpha_i-\hat\alpha_i)^2 d \sim \frac{d^3}{n}$

$\epsilon = \|\rho-\hat\rho\|_1 \le \sqrt{d}\,\|\rho-\hat\rho\|_2 \lesssim \frac{d^2}{\sqrt n} \rightarrow n \sim \frac{d^4}{\epsilon^2}$

single-copy mmts can achieve $n \sim d^3/\epsilon^2$ (also later)

why is $\frac{d^2}{\epsilon^2}$ necessary?

Lemma $\exists$ unitaries $U_1,\dots,U_M \in \mathcal{U}(d)$ $M = \exp(cd^2)$ $c>0$ const.

$\exists$ fixed projector Π, $\operatorname{tr}\Pi = d/2$

$$\frac{1}{2}\left\| U_x\frac{\Pi}{d/2}U_x^\dagger - U_y\frac{\Pi}{d/2}U_y^\dagger\right\|_1 \ge 1/10$$

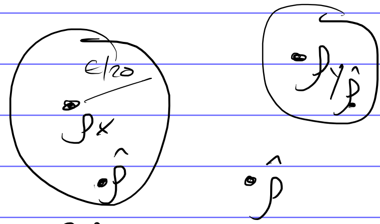
define

$$\rho_x = (1-\epsilon)\frac{I}{d} + \epsilon\, U_x\frac{\Pi}{d/2}U_x^\dagger \qquad \frac{1}{2}\|\rho_x-\rho_y\|_1 \ge \epsilon/10$$

Given a tomography procedure with accuracy $\epsilon/20$ w/ prob $1-\delta$

can determine x w/ prob $\ge 1-\delta$

$$x \longrightarrow \rho_x^{\otimes n} \longrightarrow \hat{\rho} \longrightarrow \hat{x}$$



$$Q^n = Q_1 \ldots Q_n$$

$$I(X;\hat{X}) \le I(X;Q^n) \le n\, I(X;Q_1)$$

$$I(X;\hat{X}) \ge (1-\delta)\log M - 1 \sim c' d^2$$

$$I(X;Q_1) = \underbrace{S\Big(\frac{1}{M}\sum_x \rho_x\Big)}_{\le \log d} - \frac{1}{M}\sum_{x\in[M]} S(\rho_x)$$

$$\le \log d - (\log d - O(\epsilon^2))$$

$$\sim \epsilon^2$$

$$\mathrm{eig}(\rho_x) = (1-\epsilon)\begin{pmatrix} 1/d \\ \vdots \\ 1/d \end{pmatrix} + \epsilon \begin{pmatrix} 2/d \\ \vdots \\ 2/d \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \mathrm{uniform}_{d/2} \otimes \begin{pmatrix} \frac{1+\epsilon}{2} \\ \frac{1-\epsilon}{2} \end{pmatrix}$$

$$\log d/2 + 1 - O(\epsilon^2)$$

$$H_2\Big(\frac{1-\epsilon}{2}\Big) = -\frac{1-\epsilon}{2}\log\frac{1-\epsilon}{2} - \frac{1+\epsilon}{2}\log\frac{1+\epsilon}{2}$$

$$\overset{\text{Taylor}}{=} 1 - \frac{\epsilon^2}{2\ln 2} + \cdots$$

$$\underline{\text{OR}} \quad 1 - H_2\Big(\frac{1-\epsilon}{2}\Big) = S\left(\begin{pmatrix} \frac{1-\epsilon}{2} \\ \frac{1+\epsilon}{2} \end{pmatrix} \middle\| \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}\right) \ge \frac{(\text{trace dist})^2}{2\ln 2} = \frac{\epsilon^2}{2\ln 2}$$

all together

$$c' d^2 \lesssim \frac{n\epsilon^2}{2\ln 2} \implies n \gtrsim \frac{d^2}{\epsilon^2}$$

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product measurements

replace $I(X;Q_1)$ with $I_{acc} = \max\limits_{\text{measurements from } Q\to Y} I(X;Y)$

<u>incomplete proof</u>

consider measurements in std. basis $|1\rangle, |2\rangle, \ldots, |d\rangle$

ρ_x as above

$$\langle i|\rho_x|i\rangle = \frac{1-\epsilon}{d} + \frac{\epsilon}{d/2}\langle i|U_x \Pi \underbrace{U_x^\dagger |i\rangle}_{\text{random unit vector in } \mathbb{C}^d}$$

$$U_x^\dagger|i\rangle = (a_1 + ib_1)|1\rangle + \cdots + (a_d + ib_d)|d\rangle$$

each $a_i, b_i \sim N(0, \frac{1}{2d})$ $\quad Z = \sum_{j=1}^{d/2} a_j^2 + b_j^2 \quad \mathbb{E}Z = \frac{1}{2} \quad \mathrm{Var}(Z) = d\cdot \mathrm{Var}(a_i^2) \sim 1/d$

$$P_y \approx \frac{1}{d}\begin{pmatrix} 1 \pm O(\epsilon/\sqrt{d}) \\ \vdots \\ 1 \pm O(\epsilon/\sqrt{d}) \end{pmatrix} \Rightarrow I(X;Y) \sim \frac{\epsilon^2}{d}$$