Sensing

$H = \frac{B}{2} Z$ for 1 qubit or $H = \frac{B}{2} \sum_{i=1}^{n} Z_i$

$$e^{-iHt}|+\rangle = \frac{e^{-\frac{iBt}{2}}|0\rangle + e^{\frac{iBt}{2}}|1\rangle}{\sqrt{2}} \qquad \phi = Bt$$

$$p = |\langle + | e^{-iHt} | + \rangle|^2 = \cos^2(\phi/2)$$

outcome $X = \pm 1$

$$\mathbb{E}X = \cos^2(\phi/2) - \sin^2(\phi/2) = \cos(\phi)$$

$$\operatorname{Var} X = 1 - (\mathbb{E}X)^2 = \sin^2\phi \qquad SD(X) = \sin(\phi) \equiv \Delta X$$

N repetitions $\rightarrow$ $\hat{X} \approx X \pm \frac{\sin\phi}{\sqrt{N}}$ $\qquad \hat{\phi} = \cos^{-1}(\hat{X})$

$$\hat{X} = \cos(\hat{\phi})$$

$$\frac{d\hat{X}}{d\hat{\phi}} = -\sin\hat{\phi} \qquad \frac{d\hat{\phi}}{d\hat{X}} = \frac{-1}{\sin\hat{\phi}} \qquad \Delta\phi = \frac{\Delta X}{\sin\phi} = \frac{1}{\sqrt{N}}$$ independent of ϕ

$$\Delta B = \frac{1}{t\sqrt{N}}$$

$\frac{1}{\sqrt{N}}$ scaling called the SQL (standard quantum limit)

or shot-noise limit

Is this the best we can do with N qubits?

No!

$$e^{-iHt}\frac{|0^N\rangle + |1^N\rangle}{\sqrt{2}} = \frac{e^{-\frac{iBtN}{2}}|0^N\rangle + e^{\frac{iBtN}{2}}|1^N\rangle}{\sqrt{2}}$$

$\Delta B = \frac{1}{tN}$ called the Heisenberg limit can quantify how much information a state carries about a parameter via the Fisher Information.

classical case

unknown parameter θ $\to$ observation x via $p_\theta(x)$

goal output $\hat{\theta} \approx \theta$

simpler goal distinguish p_θ from p_0 for small θ

likelihood ratio test $W_n = \log \frac{p_\theta(x_1)\cdots p_\theta(x_n)}{p_0(x_1)\cdots p_0(x_n)}$

$$\mathop{\mathbb{E}}_{x\sim p_\theta} W_n = n\, D(p_\theta \| p_0) = n\Big(\underbrace{0}_{D(p_0\|p_0)=0} + \underbrace{0\cdot\theta}_{\text{since } D\geq 0} + \underbrace{\tfrac{F}{2}\theta^2}_{\text{def of } F} + \cdots\Big)$$

$$F = \partial_\theta^2 D(p_\theta \| p_0)\big|_{\theta=0}$$
$$= \sum_x \partial_\theta^2 p_\theta(x) \log p_\theta(x)$$
$$= \sum_x p_\theta(x)\left(\partial_\theta \log p_\theta(x)\right)^2 = \sum_x \frac{(\partial_\theta p_\theta)^2}{p_\theta}\Big|_{\theta=0}$$

using $\sum_x \partial_\theta p_\theta(x) = 0$ repeatedly

when can we reliably distinguish p_θ^n from p_0^n?

need $\mathop{\mathbb{E}}_{p_\theta} W_n \gg \mathop{SD}_{p_\theta \text{ or } p_0}(W_n)$ note $\mathop{\mathbb{E}}_{p_0} W_n \leq 0$

$$\frac{1}{n}\mathop{\mathrm{Var}}_{p_\theta} W_n = \sum_x p_\theta(x) \underbrace{\log^2 \frac{p_\theta(x)}{p_0(x)}}_{\approx\, \theta^2 \partial_\theta \log p_\theta} - \underbrace{D(p_\theta\|p_0)^2}_{\sim \theta^4}$$

$$= \theta^2 \sum_x p_\theta(x)(\partial_\theta \log p_\theta)^2 = \theta^2 F$$

$$\frac{\mathbb{E} W_n}{SD(W_n)} = \frac{nF\theta^2/2}{\sqrt{nF\theta^2}} = \frac{\sqrt{nF}\,\theta}{2} \qquad \theta_{min} = \frac{2}{\sqrt{nF}} \text{ or } \frac{c}{\sqrt{nF}}$$

Cramer-Rao bound

$\hat{\theta}(x^n)$ is locally unbiased near θ_0 means

$$\mathbb{E}[\hat{\theta}] = \theta + O(\theta-\theta_0)^2$$

set $\theta_0 = 0$ for simplicity

$$\mathrm{Var}[\hat{\theta}] = \sum_{x^n} p_0(x^n)\, \hat{\theta}^2$$

$$1 = \partial_\theta \mathbb{E}[\hat{\theta}]\Big|_{\theta=0} \quad \text{(unbiased condition)}$$

$$= \partial_\theta \left(\sum_{x^n} p_\theta^n(x^n)\, \hat{\theta}(x^n) \right) = \mathbb{E}_{x^n \sim p_\theta^n} \hat{\theta}(x^n) \frac{\partial_\theta p_\theta^n(x^n)}{p_\theta^n(x^n)} = \mathbb{E}_{x^n} \hat{\theta}\, \partial_\theta \log p_\theta^n(x^n)$$

$$\langle a(x), b(x) \rangle := \mathbb{E}_{x^n} a(x^n)\, b(x^n)$$

then Cauchy-Schwarz $\Rightarrow$

$$1^2 \le \mathbb{E}\hat{\theta}^2 \cdot \mathbb{E}\left(\partial_\theta \log p_\theta^n(x^n)\right)^2$$

$$= \mathrm{Var}(\hat{\theta})\; nF$$

$$\Rightarrow \mathrm{Var}(\hat{\theta}) \ge \frac{1}{nF}$$

saturated iff $\hat{\theta} \propto \partial_\theta \log p_\theta^n$

Quantum Fisher information

Assume $\rho_\theta > 0$ $\quad \theta$ near $\theta_0 = 0$

$\mathrm{Mult}_\rho[X] = \frac{1}{2}(\rho X + X\rho)$

$\mathrm{Div}_\rho[X] = \mathrm{Mult}_\rho^{-1}[X]$ (possible when $\rho > 0$)

$L_{\rho,\theta} = \mathrm{Div}_\rho[\partial_\theta \rho] \qquad \partial_\theta \rho = \mathrm{Mult}_\rho[L] = \frac{1}{2}(\rho L + L\rho)$

$L_{\rho,\theta}$ proxy for $\partial_\theta \log \rho_\theta$

$F_Q = \mathrm{tr}[\rho L^2]$

$$\partial_\theta^2 D(\rho_\theta \| \rho_0) = \frac{F_Q \theta^2}{2} + O(\theta^3)$$

relation to classical Fisher information

$M = \{M_x\}_{x\in X} \qquad F_M \equiv F(p_\theta(x) = \mathrm{tr}\, M_x \rho_\theta) = \sum_x \mathrm{tr}\rho_\theta M_x \cdot \left(\frac{\partial_\theta \mathrm{tr}\rho_\theta M_x}{\mathrm{tr}\rho_\theta M_x}\right)^2$

$$= \sum_x \mathrm{tr}\rho_0 M_x) \quad \mathrm{Re}\left(\frac{\mathrm{tr}\,\rho_0 L M_x}{\mathrm{tr}\,\rho_0 M_x}\right)^2$$

$$\leq \sum_x \frac{|\mathrm{tr}\,\rho_0 L M_x|^2}{\mathrm{tr}\,\rho_0 M_x} = \sum_x \frac{|\mathrm{tr}\,\sqrt{\rho_0}\, L \sqrt{M_x}\sqrt{M_x}\sqrt{\rho_0}\,|^2}{\mathrm{tr}\,\rho_0 M_x}$$

$$\leq \frac{\mathrm{tr}\,(\sqrt{\rho_0}\, L\sqrt{M_x})(\sqrt{M_x}\, L\sqrt{\rho_0}) \cdot \mathrm{tr}\,\sqrt{M_x}\,\rho_0\sqrt{M_x}}{\mathrm{tr}\,\rho_0 M_x}$$

$$= F_Q$$

saturated if $\mathrm{Im}\,\mathrm{tr}\,\rho_0 L M_x = 0$

and $\sqrt{M_x}\, L\sqrt{\rho_0} \propto \sqrt{M_x}\sqrt{\rho_0}$

$L = \sum_x \lambda_x |\ell_x\rangle\langle \ell_x|$ take $M_x = |\ell_x\rangle\langle\ell_x|$ (or Π_x)

$\Rightarrow F_M = F_Q$

Q Cramer-Rao

A_n is unbiased if $\mathrm{tr}\, A_n \rho_\theta^{\otimes n} = \theta$ (optionally $+O(\theta^2)$)

$A_n = \sum_x \hat\theta_x |x\rangle\langle x| \qquad \Pr[\hat\theta_x] = \langle x|\rho_\theta^{\otimes n}|x\rangle$

$$\mathrm{Var}[\hat\theta_x] = \langle A_n^2\rangle - \langle A_n\rangle^2 \geq \frac{1}{F(x)}$$