

Randomness and concentration of measure

plan: scalar RVs, vectors, matrices

Scalar RVs

Markov's inequality: $X \geq 0 \Rightarrow \Pr[X \geq a \mathbb{E}[X]] \leq \frac{1}{a}$

weak assumptions, weak conclusions

Chebyshev's inequality: $\Pr[|X - \mathbb{E}[X]| \geq a \text{SD}(X)] \leq \frac{1}{a^2}$
apply Markov to $Y := (X - \mathbb{E}[X])^2$

could do higher moments or use moment generating functions

$$\Pr[X \geq a] = \Pr[e^{\lambda X} \geq e^{\lambda a}] \leq \frac{\mathbb{E}[e^{\lambda X}]}{e^{\lambda a}}$$

useful if $X = X_1 + \dots + X_n$ for indep R.V.s $\Rightarrow \mathbb{E}[e^{\lambda X}] = \prod_{i=1}^n \mathbb{E}[e^{\lambda X_i}]$

say each $X_i = \pm 1$ uniformly random

$$\mathbb{E}[e^{\lambda X_i}] = \cosh \lambda \leq e^{\lambda^2/2}$$

$$\Pr[X \geq n\delta] \leq \exp\left(\frac{n\lambda^2}{2} - n\lambda\delta\right) = e^{-n\delta^2/2} \quad \text{if we choose } \lambda = \delta$$

Chernoff bound

vector R.V.s

uniform distribution over $|\psi\rangle \in S^d = S(\mathbb{C}^d)$ = unit sphere

key property: invariance under $|\psi\rangle \rightarrow U|\psi\rangle$

$\forall$ measurable set S , $\forall U \in U(d)$, $\mu(S) = \mu(US)$ $\text{Unif}(S^d) = \text{Haar}(S^d)$

mild difficulty: entries aren't independent

independence and rotational symmetry $\Rightarrow$ only one choice (Herschel-Maxwell's theorem)

$$\text{Gaussian } |g\rangle = \sum_{x=1}^d g_x |x\rangle \quad g_x \sim N_{\mathbb{C}}(0, 1/d)$$

$$g_x = a_x + ib_x$$

$$a_x, b_x \sim N_{\mathbb{R}}(0, 1/(2d))$$

$$\text{density } \mu(|g\rangle) = \left(\frac{d}{\pi}\right)^d e^{-d\langle g|g\rangle}$$

$$\mathbb{E}[|g\rangle] = 0$$

$$\mathbb{E}[|g\rangle\langle g|] = \mathbb{I}/d$$

$$|u\rangle = \frac{|g\rangle}{\sqrt{2g|g\rangle}} \sim \text{Unif}(S^d)$$

inherits rotational invariance
now with unit vectors

$$\mathbb{E}[|u\rangle] = 0 \quad \mathbb{E}[|u\rangle\langle u|] = \mathbb{I}/d$$

Higher moments?

$$\mathbb{E}[|u\rangle^{\otimes k} \langle u|^{\otimes l}] = 0 \quad \text{if } k \neq l$$

$$\mathbb{E}[|u\rangle\langle u| \otimes |u\rangle\langle u|] ?$$

start with $\mathbb{E}[|g\rangle\langle g| \otimes |g\rangle\langle g|] = \sum_{\substack{x_1, x_2, \\ y_1, y_2}} \mathbb{E}[g_{x_1} g_{x_2} g_{y_1}^* g_{y_2}^*] |x_1\rangle\langle x_2| \otimes |y_1\rangle\langle y_2|$

Wick's thm:

$$\mathbb{E}[g_{x_1} g_{x_2} g_{y_1}^* g_{y_2}^*] = \mathbb{E}[g_{x_1} g_{y_1}^*] \mathbb{E}[g_{x_2} g_{y_2}^*] + \mathbb{E}[g_{x_1} g_{y_2}^*] \mathbb{E}[g_{x_2} g_{y_1}^*] = \frac{\delta_{x_1 y_1} \delta_{x_2 y_2} + \delta_{x_1 y_2} \delta_{x_2 y_1}}{d^2}$$

in general

$$\mathbb{E}[g_{x_1} \dots g_{x_n} g_{y_1}^* \dots g_{y_n}^*] = \frac{1}{d^n} \sum_{\pi \in S_n} \prod_{i=1}^n \delta_{x_i, y_{\pi(i)}}$$

$$\mathbb{E}[|g\rangle\langle g|^{\otimes n}] = \frac{1}{d^n} \sum_{\pi \in S_n} P_{\pi} \quad P_{\pi} = \sum_{x_1, \dots, x_n} |x_1, \dots, x_n\rangle\langle x_{\pi(1)}, \dots, x_{\pi(n)}|$$

Now back to $|u\rangle$.

Let $|g\rangle = r|u\rangle$ $r = \sqrt{2g|g\rangle}$ is indep. of $|u\rangle$

by rotation invariance

$$\mathbb{E}[|g\rangle\langle g|^{\otimes n}] = \mathbb{E}[r^{2n}] \mathbb{E}[|u\rangle\langle u|^{\otimes n}]$$

scalar trace 1

$$\Rightarrow \mathbb{E}[|u\rangle\langle u|^{\otimes n}] = \frac{\sum_{\pi} P_{\pi}}{\text{tr} \sum_{\pi} P_{\pi}}$$

How to interpret? First try $n=2$. $\mathbb{E}[|u\rangle\langle u|^{\otimes 2}] = \frac{\mathbb{I} + F}{d^2 + d}$

$F = \text{SWAP}$ has evs ± 1 with degeneracy $d(d \pm 1)/2$

$$\mathbb{E}[|u\rangle\langle u|^{\otimes 2}] = \frac{\frac{1}{2}(\mathbb{I} + F)}{\frac{1}{2}d(d+1)} \quad \frac{1}{2}(\mathbb{I} + F) =: \Pi \quad \text{projector onto } +1 \text{ eigenspace of } F$$

Symmetric subspace

Now general n :

$$\text{Sym}^n \mathbb{C}^d = \left((\mathbb{C}^d)^{\otimes n} \right)^{S_n} = \left\{ |\psi\rangle \in (\mathbb{C}^d)^{\otimes n} : P_\pi |\psi\rangle = |\psi\rangle \quad \forall \pi \in S_n \right\}$$

$\Pi_{\text{sym}}^{(n,d)}$ projects onto $\text{Sym}^n \mathbb{C}^d$

claim $\Pi = \Pi_{\text{sym}}^{(n,d)} = \frac{1}{n!} \sum_{\pi \in S_n} P_\pi$

more generally