Moments of random states

$$\mathbb{E}_{\substack{|u\rangle \in \mathbb{C}^d \\ \text{unit vec}}} |u\rangle\langle u|^{\otimes n} = \frac{\Pi_{sym}}{\operatorname{tr}\Pi_{sym}} = \frac{\frac{1}{n!}\sum_{\pi\in S_n} P_\pi}{\binom{d+n-1}{n}} = \frac{\sum_{\pi\in S_n} P_\pi}{d(d+1)\cdots(d+n-1)}$$

$$\Pi_{sym} = \sum_{\text{types } t} |T_t\rangle\langle T_t|$$

$$\mathbb{E}\, |g\rangle\langle g|^{\otimes n} = \frac{\sum_\pi P_\pi}{d^n} \qquad \text{(Wick's thm)}$$

e.g. $$\mathbb{E}\,|\langle u|0\rangle|^{2k} = \frac{1}{\binom{d+k-1}{k}} \qquad \begin{matrix} k=1 & 1/d \\ k=2 & \frac{2}{d(d+1)} \end{matrix}$$

Application

"anticoncentration"

$$\mathbb{E}_\psi \sum_{x\in\{0,1\}^n} |\langle\psi|x\rangle|^4 = \operatorname{tr} \sum_x |x\rangle\langle x| \otimes |x\rangle\langle x| \; \Pi_{sym} \cdot \frac{2}{2^n(2^n+1)} = \frac{2}{2^n+1} \approx \frac{2}{2^n} \quad \text{vs } \frac{1}{2^n} \text{ for uniform}$$

Rènyi entropies

$$H_\alpha(p) = \frac{1}{1-\alpha} \log \sum_x p(x)^\alpha \qquad S_\alpha(\rho) = \frac{1}{1-\alpha}\log \operatorname{tr}\rho^\alpha$$

so $H_\alpha(\text{unif}_d) = \log d$

properties

$\frac{d}{d\alpha} H_\alpha \le 0$ Schur-concave

$0 \le H_\alpha \le \log d$

pure $\quad I/d$

$\alpha < 1$ sensitive to low evalues

$\alpha > 1$ " " high evalues

$S_0 = \log \operatorname{rank}\rho$

$S_1 = S(\rho)$

$S_2 = -\log \operatorname{tr}\rho^2$

$S_\infty = -\log \|\rho\| \ge \frac{1}{2} S_2(\rho)$

$p(x) = |\langle x|\psi\rangle|^2$

$H(p) \ge H_2(p) \approx n-1$

more precisely

$$\mathbb{E}\, H_2(x) = \mathbb{E} -\log \sum_x p(x)^2 \ge -\log \mathbb{E}\, p(x)^2 \approx n-1$$

relevant for q. supremacy via random circuit sampling similarly, most pure states are very entangled

$|\psi\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ random $\left(\text{e.g. } d_A = 2^{n_A},\ d_B = 2^{n_B}\right)$

$$\mathbb{E}\, S(A) \geq \mathbb{E}\, S_2(A) = -\mathbb{E} \log \operatorname{tr} \psi_A^2 \geq -\log \mathbb{E} \operatorname{tr} \psi_A^2$$

replica trick: $\operatorname{tr} F(X \otimes Y) = \operatorname{tr} XY$

$$\operatorname{tr} \psi_A^2 = \operatorname{tr}(\psi_A \otimes \psi_A) F = \operatorname{tr} (\psi_{A_1B_1} \otimes \psi_{A_2B_2})(F_{A_1A_2} \otimes I_{B_1B_2})$$

$$\mathbb{E} \operatorname{tr} \psi_A^2 = \operatorname{tr} \mathbb{E}(\psi_{A_1B_1} \otimes \psi_{A_2B_2})(F_{A_1A_2} \otimes I_{B_1B_2})$$
$$= \operatorname{tr} \frac{I + F_{A_1A_2} F_{B_1B_2}}{D(D+1)} \cdot F_{A_1A_2} = \frac{d_A d_B^2 + d_A^2 d_B}{d_A d_B (d_A d_B + 1)} = \frac{d_A + d_B}{d_A d_B + 1} \approx \frac{1}{d_A} + \frac{1}{d_B}$$

$$\mathbb{E}\, S(A) \geq -\log\left(\frac{1}{d_A} + \frac{1}{d_B}\right)$$

$d_A = d_B \Rightarrow \mathbb{E}\, S(A) \geq \log d_A - 1$

$d_B \gg d_A \quad \mathbb{E}\, S(A) \geq \log d_A - \frac{d_A}{d_B}$

$$\left\| \psi_A - \frac{I}{d_A} \right\|_1^2 \leq d_A \left\| \psi_A - I/d_A \right\|_2^2 = d_A \operatorname{tr}(\psi_A - I/d_A)^2$$
$$= d_A \operatorname{tr} \psi_A^2 - 1 \approx \frac{d_A}{d_B}$$

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group representations $\quad R : G \to U(V)$ s.t. $R(gh) = R(g)R(h)$

G = a group, $U(V)$ = unitary operators on a vector space V

V^G = invariant subspace $= \{ |\psi\rangle \in V : R(g)|\psi\rangle = |\psi\rangle \ \forall g \in G \}$

$$\Pi = \frac{1}{|G|} \sum_{g \in G} R(g)$$

claim $\Pi = \operatorname{proj} V^G$

lemma $\quad R(h)\Pi = \Pi$

pf $\quad R(h) \sum_g R(g) = \sum_g R(hg) = \sum_g R(g)$ since $g \to hg$ is 1-1 (or measure-preserving if G is a compact group)

pf $\quad \Pi^\dagger \Pi = |G|^{-1} \sum_h \underbrace{R(h)^\dagger}_{= R(h^{-1})} \Pi = \Pi \Rightarrow \Pi$ is a projector

$R(h)\,\Pi|\psi\rangle = \Pi|\psi\rangle \Rightarrow \Pi|\psi\rangle \in V^G$

conversely if $|\psi\rangle \in V^G$ then $\Pi|\psi\rangle = \frac{1}{|G|} \sum_g \cancelto{1}{R(g)} |\psi\rangle = |\psi\rangle$