Understanding general k involves <u>representation theory</u>

$R: G \to L(V)$ or $R: G \to U(V)$ is a representation if

$R(gh) = R(g)R(h)$

e.g. $U \to U^{\otimes n}$ $\quad z \in \mathbb{Z}_p \to \omega^{yz}$ $\quad y \in \mathbb{Z}_p$ $\quad \omega = e^{2\pi i/p}$

$\pi \to P_\pi^{(d)}$

$g \to 1$

(R_1, V_1), (R_2, V_2) are equivalent if $\exists\ T \in L(V_1, V_2)$ s.t. T invertible

$$T R_1(g) = R_2(g) T \quad \forall g$$

Fact
G compact or finite $\Rightarrow$ any rep $\cong$ unitary rep.

(R,V) reducible if $(R,V) \cong (R_1,V_1) \oplus (R_2,V_2)$

$$R(g) = \left(\begin{array}{c|c} R_1(g) & 0 \\ \hline 0 & R_2(g) \end{array}\right) \quad \text{in some basis.}$$

If not (R,V) is an irrep

$\hat{G} = \{\text{inequivalent irreps}\}$

<u>Isotypic decomposition</u>

$M_\lambda \geq 0$ is the multiplicity

for any f.d. rep V, $\quad V \cong \bigoplus_{\lambda \in \hat{G}} V_\lambda \otimes \mathbb{C}^{M_\lambda}$

Regular representations

$\mathbb{C}[G] = \text{span}\{|g\rangle : g \in G\}$ (can also think of this as functions

$L(x)|g\rangle = |xg\rangle \qquad R(x)|g\rangle = |gx^{-1}\rangle \qquad f: G \to \mathbb{C}$)

both reducible

$$\mathbb{C}[G] \underset{L(g_1)R(g_2)}{\cong} \bigoplus_{\lambda \in \hat{G}} V_\lambda \otimes V_\lambda^* \underset{L}{\cong} \bigoplus_\lambda V_\lambda \otimes \mathbb{C}^{\dim V_\lambda}$$

(R^*, V^*)? $\quad R^*(g) = R(g^{-1})^T$ $\qquad \Rightarrow |G| = \sum_\lambda (\dim V_\lambda)^2$

$L(V,W) \cong V \otimes W^*$

(r,V) and (s,W) $\rightarrow$ $r^*(g) \otimes s(g)$ acting on $L(V,W)$

as $M \in L(V,W) \rightarrow s(g) M r(g)^{-1}$

e.g. $\rho \rightarrow U\rho U^\dagger \cong U \otimes U^*$

~~[illegible]~~ 1D invariant subspace $|\Phi\rangle$
the ~~[illegible]~~ d^2-1 orthogonal space
is irreducible

$U \otimes U$ has ~~[illegible]~~ two irreps V_{anti} & V_{sym}

~~[illegible]~~
In general $(U^{\otimes n}, \mathrm{Sym}^n \mathbb{C}^d)$ is an irrep of $U(d)$

Pf sketch suppose $|\psi_1\rangle, |\psi_2\rangle \in \mathrm{Sym}^n \mathbb{C}^d$

$\mathbb{E}\, |\phi\rangle\langle\phi|^{\otimes n} \propto \Pi_{sym} \Rightarrow \exists |\phi_1\rangle, |\phi_2\rangle \in \mathbb{C}^d$ s.t.

$\langle\phi_1|^{\otimes n}|\psi_1\rangle \neq 0 \qquad \langle\phi_2|^{\otimes n}|\psi_2\rangle \neq 0$

$\cdots \Rightarrow \exists\, U$ s.t. $\langle\psi_1| r(U) |\psi_2\rangle \neq 0$

Schur's Lemma V_μ, V_ν are irreps of G over $\mathbb{C}$

$L(V_\mu, V_\nu)^G$ = G-invariant maps from $V_\mu \rightarrow V_\nu$

$= \begin{cases} 0 & \text{if } \mu \neq \nu \\ \mathbb{C} I & \text{if } \mu = \nu \end{cases}$

Pf $T \in L(V_\mu, V_\nu)^G$ $\ker T \subseteq V_\mu$ and $\mathrm{Im}\, T \subseteq V_\nu$ are G-invariant subspaces

V_μ, V_ν irreps $\Rightarrow$ $\ker T = \{0\}$ or V_μ
invertible ~~[illegible]~~ 0
$\mathrm{Im}\, T = V_\nu$ or $\{0\}$

$\mu = \nu$. ~~[illegible]~~ λ = evl of T
$\ker(\lambda I - T) \neq 0$ so $= V_\mu$
$\Rightarrow T = \lambda I$

$\Rightarrow$ avg over $U^{\otimes n}$ $\Rightarrow \frac{\Pi_{sym}}{\mathrm{tr}\, \Pi_{sym}}$ (for example)