Schur-Weyl duality

$q_n(U) = U^{\otimes n}$ $\qquad P_d(\pi) = P_\pi^{(d)}$

$[q_n(U), p_d(\pi)] = 0$

this is the statement of Schur-Weyl duality

$$(\mathbb{C}^d)^{\otimes n} \underset{\substack{U_d \times S_n \\ P_d\, q_n}}{\cong} \bigoplus_{\lambda \in \mathrm{Par}(n,d)} Q_\lambda \otimes P_\lambda \quad (\otimes \mathbb{C}^1)$$

$\mathrm{Par}(n,d) = \{\lambda \in \mathbb{Z}^d : \lambda_1 \geq \cdots \lambda_d \geq 0,\ \sum_i \lambda_i = n\}$

$A = \mathrm{Span}\{q^n(U) : U \in U_d\}$ $\qquad B = \mathrm{span}\{P_\pi : \pi \in S_n\}$

$\mathrm{Comm}(A) := \{X : [X,a] = 0\ \forall a \in A\} = B$

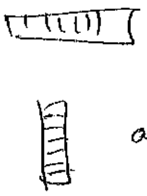
$\mathrm{Comm}(B) = A = \mathrm{Span}\{X^{\otimes n} : X \in M_d\}$

Double Commutant thm

$n=1$ $\qquad n=2$ $\qquad n=3$

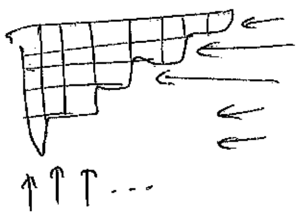
$\lambda = (1)$ $\qquad \lambda = (2)$ $\quad (1,1)$ $\qquad \lambda = (3)$ $\quad (2,1)$ $\quad (1,1,1)$ $\qquad$ etc..

symmetrize $\quad$ antisymmetrize



Sym

antisym



$\leftarrow$ symmetrize rows

antisymmetrize columns

$\#\,\mathrm{Par} \sim n^d$

$\dim Q_\lambda^d \sim n^{d^2}$ $\qquad$ if d fixed $\quad n \to \infty$

$\dim P_\lambda \sim \exp(n\, H(\lambda/n))$ $\qquad$ quantum analogue of types