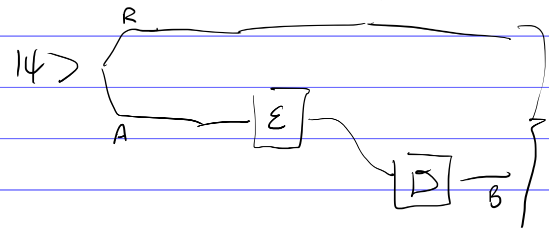
Merging: application of random unitaries to q. info. theory

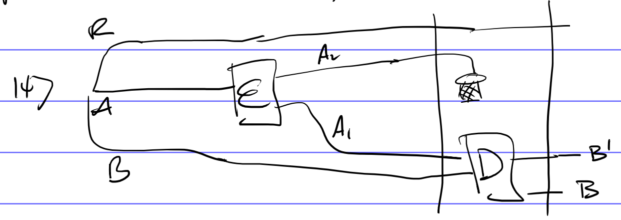
recall Schumacher compression



$\sigma^{RB} \approx \psi^{R(A\to B)}$

A→B means it's controlled by B but looks like the original A system

compression with side information



σ $\quad \rho^{RB'B} \approx \psi^{R(A\to B)B}$

Strategy

$$A = A_1 \otimes A_2 \qquad d_A = d_{A_1} d_{A_2}$$

$$\mathcal{E} = \text{random unitary } U^{A\to A_1A_2} \sim \text{Haar (or 2-design)}$$

decoupling: $\sigma^{A_2R} \approx \sigma^{A_2}\otimes\sigma^R \quad (\ast)$

purify ↓ $(U\otimes I_{BR})|\psi\rangle$ ↘ $|\varphi\rangle^{A_2\tilde{B}} \otimes |\psi\rangle^{RB'B}$

Uhlmann $\Rightarrow \exists\, V: A_1B \to \tilde{B}B'B$ s.t.

$$V_{A_1B\to\tilde{B}B'B}\, U_{A\to A_1A_2}|\psi\rangle \approx |\varphi\rangle^{A_2\tilde{B}}\, |\psi\rangle^{RB'B}$$

↑ $\mathcal{D}$ ↑ $\mathcal{E}$ ↑ bonus entanglement produced ↑ merging

Pf of $(\ast)$

First calculate $\mathbb{E}_U \|\sigma^{A_2R} - \sigma^{A_2}\otimes\sigma^R\|_2^2 = \mathbb{E}_U\left[\operatorname{tr}\sigma_{A_2R}^2 - 2\operatorname{tr}\sigma_{A_2R}(\sigma_{A_2}\otimes\sigma_R) + \operatorname{tr}\sigma_{A_2}^2\operatorname{tr}\sigma_R^2\right]$

$$\begin{aligned}
\mathbb{E}\operatorname{tr}\sigma_{A_2R}^2 &= \mathbb{E}\operatorname{tr}(\sigma_{A_2R}\otimes\sigma_{A_2'R'})F_{A_2}F_R \\
&= \mathbb{E}\operatorname{tr}(U\psi_{AR}U^\dagger\otimes U\psi_{A'R'}U^\dagger)F_{A_2}F_R \\
&= \operatorname{tr}(\psi_{AR}\otimes\psi_{A'R'})\, \underbrace{\mathbb{E}\,(U^\dagger\otimes U^\dagger)F_{A_2}(U\otimes U)}_{=\ \alpha_+\Pi_+ + \alpha_-\Pi_-} \otimes F_R \\
&= \frac{\alpha_+ + \alpha_-}{2} I + \frac{\alpha_+ - \alpha_-}{2} F_A \approx d_{A_2}^{-1} I + d_{A_1}^{-1} F_A
\end{aligned}$$

$$\Rightarrow \mathbb{E}\operatorname{tr}\sigma_{A_2R}^2 \approx d_{A_2}^{-1}\operatorname{tr}\psi_R^2 + d_{A_1}^{-1}\operatorname{tr}\psi_{AR}^2$$

$$\Pi_\pm = \frac{I \pm F_{A_1}F_{A_2}}{2} \qquad \alpha_\pm = \frac{\operatorname{tr}\Pi_\pm F_{A_2}}{d_A(d_A\pm 1)/2} \approx \frac{1}{d_{A_2}} \pm \frac{1}{d_{A_1}}$$

$$\begin{aligned}
&\operatorname{tr}(I \pm F_{A_1A_2})F_{A_2} \\
&= \operatorname{tr} F_{A_2} \pm F_{A_1} \\
&= d_{A_1}^2 d_{A_2} \pm d_{A_1} d_{A_2}^2 = d_A(d_{A_1} \pm d_{A_2})
\end{aligned}$$

$$\mathbb{E}\ \mathrm{tr}\,\sigma_{A_2}^2\ \mathrm{tr}\,\sigma_R^2 = \mathbb{E}\left(\mathrm{tr}\,\sigma_{A_2}^2\right)\cdot \mathrm{tr}\,\psi_R^2$$

$$\mathbb{E}\ \mathrm{tr}\,\sigma_{A_2}^2 = \mathrm{tr}\ \psi_{AR}\otimes\psi_{A'R'}\ \mathbb{E}\left(U^\dagger\otimes U^\dagger\right) F_{A_2}(U\otimes U) \approx d_{A_2}^{-1} + d_{A_1}^{-1}\mathrm{tr}\,\psi_A^2 \approx d_{A_2}^{-1}$$

if $d_{A_1} > d_{A_2}$

$$\mathbb{E}\ \mathrm{tr}\,\sigma_{A_2R}\left(\sigma_{A_2}\otimes\sigma_R\right) = \mathbb{E}\ \mathrm{tr}\,(U\otimes U)(\psi_{AR}\otimes\psi_{A'}\otimes\psi_{R'})(U^\dagger\otimes U^\dagger)F_A F_R$$

$$= \mathrm{tr}\,(\psi_{AR}\otimes\psi_{A'}\otimes\psi_{R'})\left(d_{A_2}^{-1} F_R + d_{A_1}^{-1} F_A F_R\right)$$

$$= d_{A_2}^{-1}\ \mathrm{tr}\,\psi_R^2 + d_{A_1}^{-1}\ \mathrm{tr}\,\psi_{AR}(\psi_A\otimes\psi_R)$$

$d_{A_2}^{-1}\ \mathrm{tr}\,\psi_R^2$ terms cancel

$$\Rightarrow \mathbb{E}\ \|\sigma_{A_2R} - \sigma_{A_2}\otimes\sigma_R\|_2^2 \approx d_{A_1}^{-1}\left(\mathrm{tr}\,\psi_{AR}^2 - 2\,\mathrm{tr}\,\psi_{AR}(\psi_A\otimes\psi_R) + \mathrm{tr}\,\psi_A^2\psi_R^2\right)$$

$\exists U$ s.t. $\|\cdot\|_2^2 \leq \ldots$

fix this U.

$$\|\sigma_{A_2R} - \sigma_{A_2}\otimes\sigma_R\|_1^2 \leq \frac{d_{A_2}d_R}{d_{A_1}}\left(\mathrm{tr}\,\psi_{AR}^2 + \mathrm{tr}\,\psi_A^2\,\mathrm{tr}\,\psi_R^2\right)$$

Is this small? yes for $\frac{d_{A_1}}{d_{A_2}}$ big enough.

concretely

$$\rho_{AB} \xrightarrow{\text{purify}} |\phi\rangle_{ABR} \rightarrow |\psi\rangle = c\left(\Pi^n_{\phi_{A,\delta}}\otimes\Pi^n_{\phi_{B,\delta}}\otimes\Pi^n_{\phi_{R,\delta}}\right)|\phi\rangle^{\otimes n}$$

$$\mathrm{tr}\,\psi_A^2 \approx \exp(-n\,S(A)_\phi) = \exp(-n\,S(A)_\rho)$$

$$\mathrm{tr}\,\psi_R^2 \approx \exp(-n\,S(R)_\phi) = \exp(-n\,S(AB)_\rho)$$

$$\mathrm{tr}\,\psi_{AR}^2 \approx \exp(-n\,S(AR)_\rho) = \exp(-n\,S(B)_\rho) \geq \exp(-n(S(A)+S(R)))$$

$$d_R \approx \exp(n\,S(AB)_\rho) \qquad d_A \approx \exp(n\,S(A)_\rho)$$

error small if $\quad d_A d_R\ \mathrm{tr}\,\psi_{AR}^2 \ll d_{A_1}^2$

$$\log d_{A_1} \gg n\,\frac{I(A;R)_\phi}{2}$$

yields $\log d_{A_2}$ ebits $\qquad \log d_{A_2} \approx n\left(S(A) - \frac{I(A;R)}{2}\right) = n\,\frac{I(A;B)}{2}$

$$\langle\phi^{ABR}\rangle + \frac{I(A;R)}{2}\ \underset{\longrightarrow}{\text{qubits}} \geq \langle\phi^{(A\to B)BR}\rangle + \frac{I(A;B)}{2}\ \text{ebits}$$

If cbits are free then the entanglement cost is

$$\frac{I(A;R) - I(A;B)}{2} = \frac{1}{2}\left(S(A)+S(R)-S(B) - \left(S(A)+S(B)-S(R)\right)\right) = S(R) - S(B)$$

$$= S(AB) - S(B) = S(A|B)$$