Topological order

Toric code has distance L.

C = code space $\quad \dim C = 4$

Let M be an operator with diam supp $M < L/2$

then $\Pi_C M \Pi_C \propto \Pi_C$

In general, Π has (ℓ, ϵ) topological order if

diam supp $M < \ell \Rightarrow \min_z \| \Pi M \Pi - z\Pi \| \leq \epsilon \|M\|$

need $\dim \Pi \geq 2$

Suppose $|\psi_1\rangle, |\psi_2\rangle$ (ie $\Pi = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|$) have (ℓ, ϵ) topo order

$|\phi_1\rangle := e^{-iHt}|\psi_1\rangle \qquad |\phi_2\rangle = e^{-iHt}|\psi_2\rangle$

$|\phi_1\rangle, |\phi_2\rangle$ have $(\ell - vt, \epsilon + \delta)$ for $v > v_{LR}$

$\Rightarrow$ toric code states need time $\Omega(L)$ to prepare from local Hamiltonians

AGSP

L-R $\quad \exp(-i\,\mathrm{ad}_H t)$ is approximately $v_{LR} t$-local

$v_{LR} \lesssim \max_i \sum_j \|h_{ij}\|$ for 2-local H

$\int \exp\left(\frac{-t^2}{2T^2}\right) \exp(-i\,\mathrm{ad}_H t)\, dt$ is approx $v_{LR} T$-local

and $\approx$ projects onto energies in $O(T^{-1})$-size window

H^k is exactly k-local

ie $\quad \mathrm{supp}\, [H^k, A_X] \subseteq X_k$ = k-fattening of X

$$|gs\rangle\langle gs| = \lim_{k\to\infty} \left(\frac{E_0 - H}{E_{max} - E_0}\right)^k$$

starts to converge when $k \sim \frac{\|H\|}{g}$

say eigs$(H) = 0, g, \ldots, 1$

eig$(I - H)^k = 1, (1-g)^k, \ldots, 0$

take $k = \frac{\log 1/\epsilon}{g}$

unfortunately $\|H\| \sim n$ typically

Improvement using Chebyshev polys

$p(x) = x^k$ has $p(1) = 1$ $\quad p([-1,1]) \subseteq [-1,1]$

deg $p = k$ $\quad p(1-\epsilon) \approx e^{-k\epsilon}$

or $1-k\epsilon$

we can do better!

$T_k(x)$ $\quad T_k(\cos\theta) = \cos(k\theta)$ $\quad 1, x, 2x^2-1, 4x^3-3x, \ldots$

$T_{k+1} = 2xT_k - T_{k-1}$ $\quad$ orthogonal w.r.t. $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$ $\quad$ (not normalized, though...)

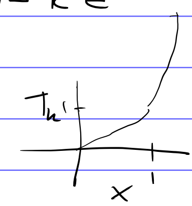
$T_k([-1,1]) = [-1,1]$

$T_k(1) = 1$ $\quad T_k(1-\epsilon) = \cos(k \arccos(1-\epsilon)) \approx \cos(k\sqrt{2\epsilon}) \approx 1-k^2\epsilon$

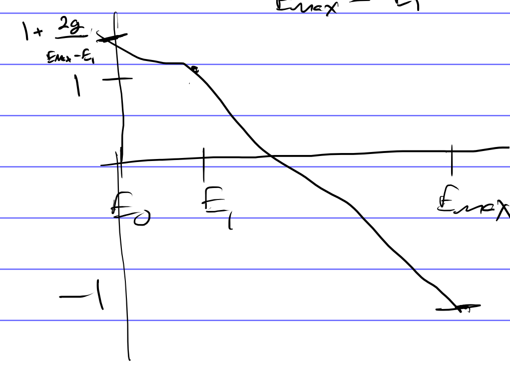
$1-\epsilon \approx 1-\theta^2/2$

$x>1$? $\quad \cos\theta = x$ $\quad \theta = it$ $\quad x = \cosh t$ $\quad T_k(x) = \cosh(kt) \geq \frac{1}{2}e^{kt}$

$t \geq 2\sqrt{\frac{x-1}{x+1}}$



Define $\quad p_k(E) = \frac{T_k(f(E))}{T_k(f(E_0))}$ $\quad f(E) = \frac{E_{max}+E_1-2E}{E_{max}-E_1}$



$E_0 = E_1 - g$

$f(E_0) = \frac{E_{max}-E_1+2g}{E_{max}-E_1}$

$p_k(E_0) = 1$

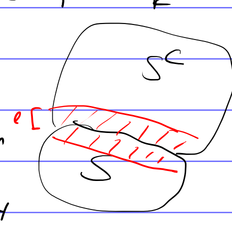
$p_k([E_1, E_{max}]) \lesssim e^{-2k\sqrt{\frac{g}{E_{max}-E_1}}} \leq e^{-2k\sqrt{\frac{g}{\|H\|}}}$ $\quad f$

Yields another way to distinguish QECC states from low-complexity states

Let U have locality k, i.e. $|\mathrm{supp}\ UAU^\dagger| \leq k \cdot |\mathrm{supp}\ A|$

$p(x) := |\langle x|\psi\rangle|^2$ $\quad |\psi\rangle = U|0^n\rangle$

$h_\ell(p) = \min_{S:\ 0<p(S)\leq 1/2} \frac{p(\partial_\ell(S))}{p(S)}$ $\quad \partial_\ell(S)$ = points within dist ℓ of the boundary



$\ell = \frac{k}{4}(kn)^{\frac{1}{2}-\gamma} \Rightarrow h_\ell(p) \geq \frac{(kn)^{-2\gamma}}{8}$

$k = O(1)$ $\quad \ell \sim \sqrt{n}$ $\quad h_\ell \geq$ const

$h_{\sqrt{n}}$(unif) $\sim$ const

rules out codes with Z distance $> \ell$, say $\frac{|0^n\rangle + |1^n\rangle}{\sqrt{2}}$

Pf $\quad R_S = I - 2\sum_{x\in S} |x\rangle\langle x|$ $\quad \langle\psi|\psi'\rangle = 1-2p(S)$ $\quad |\psi'\rangle = R_S|\psi\rangle$

construct A s.t. $\quad H_0 = \frac{1}{n}\sum_i |1\rangle\langle 1|_i$ $\quad H = UH_0U^\dagger$ k-local

$\langle\psi|A|\psi\rangle = 0$ $\quad A = p_m(H)$ $\quad m = \frac{1}{2}(kn)^{\frac{1}{2}-\gamma}$ $\quad \ell = \frac{km}{2}$

$4p(\partial_\ell(S)) \geq \langle\psi'|A|\psi'\rangle \geq \frac{1}{2}(nk)^{-2\gamma} p(S)$