Lecture 15: October 31, 2024

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In this lecture, we'll build the background for a more principled approach to these calculations.

15.0 Random Unitaries and Haar Measure

The uniform distribution over the unitary group $U(d)$ is given by the **Haar measure**, denoted μ_{Haar}. This measure is the analog of the uniform distribution for any compact group and is defined by the following properties:

- **Normalization**: $\mu_{\text{Haar}}(U(d)) = 1$.
- **Invariance**: For any subset $S \subseteq U(d)$ and any $U \in U(d)$, we have

$$\mu_{\text{Haar}}(S) = \mu_{\text{Haar}}(US) = \mu_{\text{Haar}}(SU).$$

Physicist's Perspective

This invariance implies that the measure is "uniform" in the sense that if we rotate any subset of unitaries, its measure does not change. Thus, the Haar measure provides a probability density that is independent of the choice of coordinates on $U(d)$.

The Haar measure is unique: if any probability distribution on $U(d)$ is left invariant, it must be the Haar measure. This left and right invariance essentially characterizes it, making it a fundamental tool in defining randomness for unitary operations.

15.0.1 Random Unit Vectors and Gaussian Sampling

For generating random unit vectors computationally, a common approach is to sample each component independently from a Gaussian distribution (which is rotationally invariant), and then normalize the vector. This produces a uniformly random vector on the unit sphere.

For random unitaries, one approach involves using the Gaussian Unitary Ensemble (GUE), a distribution over Hermitian matrices. To construct a GUE matrix:

- Diagonal entries are real, Gaussian-distributed.
- Off-diagonal entries are independent, with real and imaginary parts drawn from Gaussian distributions.

Given a Hermitian matrix X sampled from GUE, the unitary e^{iX} approximates a Haar-random unitary. This method is computationally feasible though not perfect, as it involves approximately $O(d^3)$ operations.

Practical Limitations

Sampling a truly Haar-random unitary is computationally challenging in high dimensions due to the large number of degrees of freedom. Approximate methods such as GUE provide feasible alternatives for many practical applications.

15.0.2 Introduction to Unitary k-designs

A **unitary k-design** is a distribution over unitaries that mimics the Haar measure up to the k-th moment. Formally, for any integer k, a distribution ν on $U(d)$ is a k-design if the following holds:

$$\mathbb{E}_{U\sim\nu}\left[U^{\otimes k,k}\right] = \mathbb{E}_{U\sim\text{Haar}}\left[U^{\otimes k,k}\right].$$

This means that, up to the k-th moment, the behavior of unitaries sampled from ν is indistinguishable from unitaries sampled according to the Haar measure.

Definition: k-design via Tensor Powers

The notation $U^{\otimes k,k}$ is shorthand for $U^{\otimes k}\otimes U^{*\otimes k}$, which represents k copies of U acting on the system along with k copies of the complex conjugate of U. A distribution ν on $U(d)$ is a k-design if choosing U from ν gives the same distribution for $U^{\otimes k,k}$ as choosing U from the Haar measure.

15.0.3 Level Repulsion and Eigenvalue Distribution

One notable feature of random unitary matrices is **level repulsion**, where eigenvalues tend to avoid being close to one another. The probability density of the eigenvalues $\{\lambda_i\}$ includes a term like

$$\prod_{i<j}|\lambda_i-\lambda_j|^2,$$

which vanishes when two eigenvalues coincide. This "repulsion" is similar to the behavior of charges repelling each other, leading to eigenvalues that spread out more evenly on the unit circle.

Visualization Exercise

Plotting the eigenvalues of a Haar-random unitary matrix on the complex unit circle reveals this level repulsion. Comparing this distribution with randomly chosen phases $e^{i\theta}$ illustrates the difference: in the Haar case, the eigenvalues push each other apart, while in the random phase case, they can cluster by chance.

This phenomenon is a well-known property in random matrix theory and also appears in GUE. When eigenvalues are distinct, each eigenvector has more degrees of freedom. For degenerate eigenvalues, the dimensionality of the associated subspace decreases.

15.0.4 Compactness and the Haar Measure

The Haar measure is defined for compact groups, such as $U(d)$, where the group has a finite total volume that can be normalized to 1. For non-compact groups, such as $SL(2,\mathbb{R})$, there is no normalized Haar measure due to infinite volume.

Fact: Invariant Subspace Projection

For a compact group G, averaging a representation R over G with the Haar measure yields a projection onto the invariant subspace V^G:

$$\int_G R(g)\, d\mu_{\text{Haar}}(g) = \text{Proj}_{V^G}.$$

This concept is crucial in representation theory and plays a foundational role in understanding k-designs.

15.0.5 Classical Analogue: k-wise Independent Hash Functions

In classical computing, k-designs have an analogy in **k-wise independent hash functions**. A hash function $h : U \to [m]$ is k-wise independent if, for any distinct inputs $x_1, \ldots, x_k$ and outputs $y_1, \ldots, y_k \in [m]$, we have:

$$\Pr_{h \in H} \left(h(x_1) = y_1 \wedge \cdots \wedge h(x_k) = y_k\right) = \frac{1}{m^k}.$$

This implies that the hash function behaves like a truly random function when viewed through any k inputs, although it is not completely random.

Construction

Degree-k polynomials are often used to construct k-wise independent hash functions. These provide a computationally efficient balance between determinism and the randomness required for various applications.