17.1 Representation Theory Continued

This lecture continues discussion of representation theory for unitary group and it's relation to random unitaries. Recall statement, claimed without proof, from the last lecture that allowed us to show that $\mathbb{E}\left|\psi\right\rangle\left\langle\psi\right|^{\otimes n} = \frac{\Pi_{sym}}{\operatorname{tr}\Pi_{sym}}$:

Lemma 17.1.1.

$$(U^{\otimes n}, \mathrm{Sym}^n\mathbb{C}^d) \textit{ is an irreducible representation (irrep) of } U(d)$$

Here $Sym^n\mathbb{C}^d$ *is a symmetric subspace:*

$$Sym^n\mathbb{C}^d = \{\left|\psi\right\rangle \in \mathbb{C}^{d^{\otimes n}} : P_\pi\left|\psi\right\rangle = \left|\psi\right\rangle \forall n \in S_n\}$$

Proof. To show that $(U^{\otimes n}, \mathrm{Sym}^n\mathbb{C}^d)$ is an irrep, we first can notice that it's clearly a representation; and symmetric subspace, when acted on by $U^{\otimes n}$, stays in symmetric subspace. Claim $(U^{\otimes n}, \mathrm{Sym}^n\mathbb{C}^d)$ is an irrep is equivalent to saying that no invariant subspace exist ($W \subset \mathrm{Sym}^n\mathbb{C}^d$: $\forall U,\ U^{\otimes n}W = W$).
If symmetric subspace was reducable and in some basis $U^{\otimes n}$ it is block-diagonal, we could have selected ψ from different blocks and for any $U^{\otimes n}$ inner product would have been 0. Conversely, the statement below is equivalent to the lemma formulation:

$$\forall\left|\psi_1\right\rangle, \left|\psi_2\right\rangle \in \mathrm{Sym}^n\mathbb{C}^d/0 \quad \exists U, \text{ s.t. } \left\langle\psi_1\right| U^{\otimes n}\left|\psi_2\right\rangle \neq 0$$

One way to show this involves using $\mathbb{E}\left|\phi\right\rangle\left\langle\phi\right|^{\otimes n} = \frac{\Pi_{sym}}{\operatorname{tr}\Pi_{sym}}$. While to prove this fact we relied on lemma 17.1.1., it can also be shown by Gaussian integration, so this isn't a circular reasoning.
If we restrict $\left|\psi_i\right\rangle = \left|\phi_i\right\rangle^{\otimes n}$, the statement becomes trivial, since it's enough to set $U\left|\phi_2\right\rangle = \left|\phi_1\right\rangle$. Symmetric subspace, however, is larger than tensor states and includes their superposition. For the next step, recall that in symmetric subspace: $\left|\psi_i\right\rangle \propto \Pi_{sym}\left|\psi_i\right\rangle \propto \mathbb{E}\left|\phi\right\rangle\left\langle\phi\right|^{\otimes n}\left|\psi_i\right\rangle$ (from here we'll ignore normalization for simplicity). Under some k-design, it's a superposition of finite tensor-product states. This tells us:

$$\exists\left|\phi_i\right\rangle, \text{ s.t. } \left\langle\psi_i\right|\left|\phi_i\right\rangle^{\otimes n} \neq 0 \quad i = \{1, 2\}$$

Without loss of generality, by selecting basis, take $\left|\phi_1\right\rangle = \left|1\right\rangle$. Now consider random unitary of form:

$$V = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & U(d-1) \end{array}\right) \Rightarrow V\left|\phi_1\right\rangle = \left|\phi_1\right\rangle$$

Then V acts on $\left|\psi_1\right\rangle$ proportionally to $\left|1\right\rangle^{\otimes n}$; and since everything else is averaged away, we can claim:

$$\mathbb{E}_V V^{\otimes n} \propto \left|\phi_1\right\rangle^{\otimes n}$$

Similarly, we can choose some W, s.t.:

$$\mathbb{E}_W W^{\otimes n} \propto |\phi_2\rangle^{\otimes n}$$

Finally, we can simply pick some U that translates $|\phi_1\rangle$ to $|\phi_2\rangle$. By averaging away all the constructed unitaries (V, W) we get:

$$\mathbb{E}_U \langle\psi_1| U^{\otimes n} |\psi_2\rangle \neq 0$$

Since average is non-zero, there certainly exists some deterministic unitary, s.t. inner product is non-zero. Therefore, symetric subspace is an irrep. □

17.2 Physics Application

Here we'll briefly discuss how symmetric and anti-symmetric subspaces relate to space of bosons (represent forces) and fermions (represent matter).
Suppose you have n bosons in d bosonic modes (e.g. harmonic oscillator). Each one of them lives in $\mathbb{C}^d$ space. Then n bosons can be described by $\mathrm{Sym}^n\mathbb{C}^d$ subspace. In contrast, n *distinguishable* particles live in space $(\mathbb{C}^d)^{\otimes n}$.
This idea is used in quantum information processing based on photons as qubits. For example, in linear optical quantum computing with n photons amount of degrees of freedom is limited by $\mathrm{Sym}^n\mathbb{C}^d$.
In opposite, n fermions in d modes stay in anti-symmetric subspace:

$$\mathrm{Anti}^n\mathbb{C}^d = \{|\psi\rangle \in \mathbb{C}^{d^{\otimes n}} : P_\pi |\psi\rangle = \mathrm{sgn}(\pi) |\psi\rangle \, \forall n \in S_n\}$$

This relates to Pauli exclusion principle - no two fermions exist in the exact same state - so that projection of tensor state onto anti-symmetric subspace is zero.
Consider special case of $n = 2$:

$$\mathbb{C}^d \otimes \mathbb{C}^d = \mathrm{Sym}^2\mathbb{C}^d \oplus \mathrm{Anti}^2\mathbb{C}^d$$

This can also be seen from the perspective of dimensions:

$$\dim \mathrm{Sym}^n\mathbb{C}^d + \dim \mathrm{Anti}^n\mathbb{C}^d = \binom{d+n-1}{n} + \binom{d}{n} = \frac{d(d+1)}{2} + \frac{d(d-1)}{2} = d^2$$

For $n = 3$, however, this is no longer true:

$$(\mathbb{C}^d)^{\otimes 3} = \mathrm{Sym}^3\mathbb{C}^d \oplus \mathrm{Anti}^3\mathbb{C}^d \oplus \text{'something else'}$$

For $d = 2$, for instance, $\mathrm{Anti}^n\mathbb{C}^d = \varnothing$, symmetric subspace has dimension 4 and corresponds to spin 3/2, therefore, the rest ('something else') must represent part with spin 1/2. Generally, ('something else') part can be described by Schur-Weyl duality.

17.3 Schur-Weyl duality

Schur-Weyl Duality is a theorem that relates the irreducible representation of Schur-Weyl Duality is a theorem that relates the irreducible representation of finite dimensional general linear group (in this part, we focus on the unitary group) and symmetric group.

Consider a tensor space with n particles:

$$\mathbb{C}^d \otimes \mathbb{C}^d \otimes ... \otimes \mathbb{C}^d \tag{17.1}$$

The two actions that can be applied on this tensor space are:

$$q_n(U) = U^{\otimes n}, \quad q_n(U)(v_1 \otimes v_2 \otimes ... \otimes v_n) = Uv_1 \otimes ... \otimes Uv_n \tag{17.2}$$

$$P_d(\pi) = P_\pi^{(d)}, \quad P_d(\pi)(v_1 \otimes v_2 \otimes ... \otimes v_n) = v_{\pi^{-1}(1)} \otimes ... \otimes v_{\pi^{-1}(n)} \tag{17.3}$$

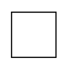
The two actions commute, $[q_n(U), P_d(\pi)] = 0$. And, the Schur-Weyl Duality asserts that:

$$(\mathbb{C}^d)^{\otimes n} \cong \bigoplus_\lambda Q_\lambda^{(d)} \otimes P_\lambda \tag{17.4}$$

where $Q_\lambda^{(d)}$ is an irreducible representation of $U(d)$, and P_λ is an irreducible representation of S_n. Generally, the decomposition of $(\mathbb{C}^d)^{\otimes n}$ can be written as $(\mathbb{C}^d)^{\otimes n} = \bigoplus_\lambda Q_\lambda^{(d)} \otimes P_\lambda \otimes \mathbb{C}^{M_\lambda}$. But specifically for the group that we are considering, Schur-Weyl duality applies.

Here, λ runs over all partition of n with d parts. $Par(n,d) = \{(\lambda_1, ..., \lambda_d) \in \mathbb{Z}^d, \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_d \geq 0, \lambda_1 + ... + \lambda_d = n\}$. Each partition can be represented by a Young diagram.

For example: $n = 1$, $\lambda = (1)$:

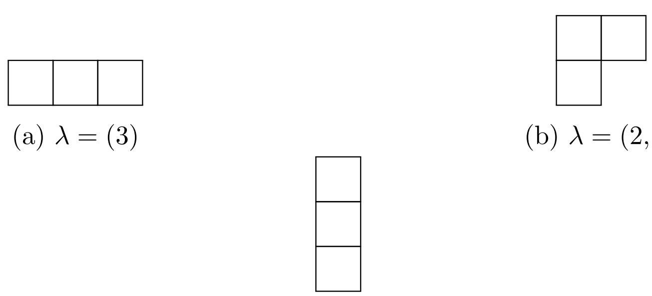


For $n = 2$, $\lambda = (2), (1,1)$

(a) $\lambda = (2)$ (b) $\lambda = (1,1)$

Figure 17.1: Young diagrams for $n = 2$ case.

For $n = 3$, $\lambda = (3), (2,1), (1,1,1)$:



(a) $\lambda = (3)$ (b) $\lambda = (2,1)$

(c) $\lambda = (1,1,1)$

Figure 17.2: Young diagrams for $n = 3$ case.

Young tableau is obtained by filling in the boxes of the Young diagram with symbols taken from some alphabet. Specifically, a tableau is called standard if the entries in each row and each column are increasing and is called semi-standard if entries weakly increase along each row and strictly increase down each column.

1	2
3	

(a) Standard Young Tableau (SYT)

1	1
2	

(b) Semi-Standard Young Tableau (SSYT)

Figure 17.3: Comparison of Two Young Tableaux for $\lambda = (2, 1)$.

The irreducible representation of S_n defines the symmetry of vectors under permutation. Basically, the vectors with labels on the same row are symmetric under permutation and anti-symmetric under permutation for vectors with labels on the same column. Similarly, the irreducible representation of $U(d)$ labeled by a given Young diagram has the symmetry of the irrep of S_n with the same Young diagram, where n is the number of boxes in the diagram. The Young diagram of an irrep of $U(d)$ has at most d rows. Therefore, basis for P_λ is indexed by SYT, since it correspond to fill in n boxes with n different alphabet words following rules of SYT. Basis for $Q_\lambda^{(d)}$ correspond to filling in n boxes with d different alphabet words following rules of SSYT. For a Young Diagram of a single row, the irrep is completely symmetric; for a Young diagram of a single column, the irrep is anti-symmetric; for a Young diagram of a mixture of rows and columns, the irrep has mixed symmetry. Therefore, for $\lambda = (n)$, $\dim(Q_\lambda) = \dim(V^n \mathbb{C}^d) = \binom{n+d-1}{n}$, while for $\lambda = (1, ..., 1)$ with $d \geq n$, $\dim(Q_\lambda) = \binom{d}{n}$.

Following the same example as above, we consider the case: $\lambda = (2, 1), n = 3, d = 2$:

1	2
3	

1	3
2	

Figure 17.4: $\dim(P_\lambda) = 2$ (SYT).

1	1
2	

1	3
2	

Figure 17.5: $\dim(Q_\lambda^{(d)}) = 2$ (SSYT).

Quantum analogue of types: For fixed d and $n \approx \infty$: $\dim(Par(n, d)) \sim n^d$. $\dim(Q_\lambda^{(d)}) \sim n^{d^2}$. $\dim(P_\lambda) \sim \exp(nH(X_n))$.