8.372 Quantum Information Science III **Fall 2024**

Lecture 18: November 7, 2024

Scribe: Ruoyi Yin *Algorithms for Semidefinite Programming*

The lecture is based on the paper arXiv: 1909.04613. The paper develops a quantum algorithm for faster semidefinite programming problem for binary quadratic optimization.

Base Problem

Problem Statement: Quadratic optimization problems with binary constraints are formulated as: for a (real-valued) symmetric $n \times n$ matrix A, compute:

$$\max \quad \langle x, Ax \rangle = \mathrm{tr}(Axx^*)$$

subject to

$$x \in \{-1, 1\}^n$$

where $x \in \mathbb{R}^n$.

This task has applications for solving many important problems, such as:

- **MaxCut:** Largest cut in a graph.
- **Community Detection:** Dividing a network into sets of nodes corresponding to two communities.

18.1 Relaxation Approach

The strategy used to speed up this optimization algorithm can be broken down into three phases:

Three Phases

- **Phase I: Relax the Problem:** Since the problem is NP-hard in the worst case, we relax it to something more manageable.
- **Phase II: Optimization Problem → Feasibility Problem:** Convert the optimization problem into a feasibility problem by formulating constraints.
- **Phase III: Quantum-Inspired Algorithm:** Develop quantum algorithm to solve the problem.

18.1.1 Phase I: Problem Relaxation and Rescaling

The set $\mathbb{S}^n$ consists of $n \times n$ positive semidefinite (PSD) matrices:

$$\mathbb{S}^n = \{X : xx* = X, X \succeq 0\}.$$

SDP Relaxation:

$$\max_{X \in \mathbb{S}^n} \quad \mathrm{tr}(AX) \text{ s.t. } X \succeq 0, \quad \mathrm{diag}(X) = \vec{1}$$

Rescaling

$$\max_{X \in \mathbb{S}^n} \quad \operatorname{tr}(\tilde{A}X), \quad \tilde{A} = \frac{1}{||A||} A \text{ s.t. } X \succeq 0, \quad \operatorname{diag}(X) = \frac{1}{n}\vec{1}, \quad \operatorname{tr}(X) = 1$$

18.1.1.1 Remark:

(i) This is a special case of convex optimization problems:

$$\max \quad f(X) = \operatorname{tr}(\tilde{A}X)$$

$X \in \mathcal{C}_1 \cap \mathcal{C}_2$, where

$$\mathcal{C}_1 = \left\{ x : \operatorname{diag}(x) = \frac{1}{n} \right\} \quad \text{affine subspace}$$

$$\mathcal{C}_2 = \{x : x \succeq 0\} \quad \text{convex cone}$$

(ii) The algorithm will work for a more general class of convex optimization problem: for a bounded, concave function $f(X)$, and $\mathcal{C}_1, ..., \mathcal{C}_n$ are closed convex sets:

$$\max \quad f(X) = \operatorname{tr}(\tilde{A}X)$$

subject to:

$$X \in \mathcal{C}_1 \cap ... \cap \mathcal{C}_n, \quad \operatorname{tr}(X) = 1, \quad X \geq 0.$$

18.1.2 Phase II: Feasibility Problem

The feasibility problem involves finding $X \in \mathbb{S}^n$ such that:

$$\operatorname{tr}(\tilde{A}X) \geq \lambda, \quad \operatorname{diag}(X) = 1, \quad \operatorname{tr}(X) = 1, \quad X \succeq 0.$$

By wrapping this task into an outer loop where we binary search the interval to choose value of λ, we only need $\log(1/\epsilon)$ queries to get multiplicative ϵ-approximation.

18.1.3 Phase III: Quantum-Inspired Change of Variable

$$X = \frac{e^{-H}}{\operatorname{tr}(e^{-H})} \in S_n \quad \text{(Gibbs state)}$$

- ensures X is PSD, trace 1.

New Problem:

$$\text{Let } \tilde{A} = \frac{A}{||A||}, \quad \text{find } H \in S^n$$

$$\text{s.t. } \operatorname{tr}(\tilde{A}\rho_H) \leq \lambda \quad (\rho_H \in \mathcal{A}_\lambda)$$

$$\operatorname{diag}(\rho_H) = \frac{I}{n} \quad (\rho_H \in \mathcal{D}_n)$$

Again: We can solve this for any number of convex constraints.

18.2 The Algorithm

18.2.1 Oracle Access

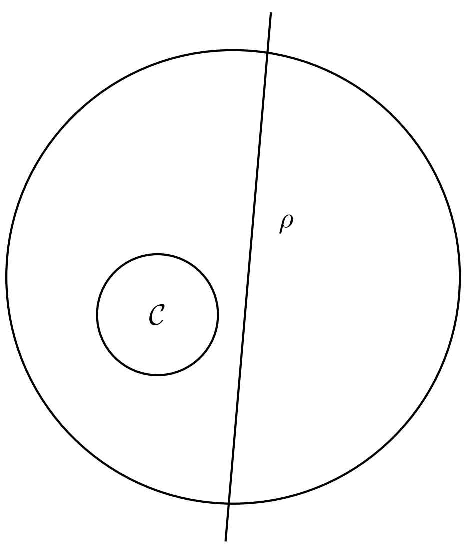
Def (ϵ-separation oracle): contains every line segment between two points in the set.
Let:

$$\mathcal{C} \subset S^n \text{ be a closed, convex subset of quantum states,}$$

$$\mathcal{C}^* = \{X = X^\dagger \in \mathbb{C}^{n\times n} : \|X\| \leq 1\} \quad \text{closed, convex subset of observables, ``tests''}$$

$$\mathcal{O}_{\mathcal{C},\epsilon}(\rho) = \begin{cases} \text{accept } \rho & \text{if } \min_{Y\in\mathcal{E}} \max_{P\in\mathcal{C}^*} \operatorname{tr}(P(\rho - Y)) \leq \epsilon \\ & \text{Interpretation: observables from } \mathcal{C}^* \text{ cannot distinguish } \rho \text{ from elements of } \mathcal{C} \\ \text{else: output} & P \in \mathcal{C}^* \text{ such that } \operatorname{tr}(P(\rho - Y)) \geq \frac{\epsilon}{2} \quad \forall Y \in \mathcal{E} \\ & \text{Interpretation: there's an observable to which } \rho \text{ looks different from all states in } \mathcal{C}. \end{cases}$$

Intuition:



($\mathcal{C}^*$ = set of planes)

If an oracle told me P, I can always improve my guess to push toward.

18.2.2 Hamiltonian Updates

Start with $H = 0$ ("infinite temperature"), $\rho_H = I/n$.
For $t = 1, ..., T$,

- check if $\rho_H \in A_\lambda$ and $\rho_H \in D_n$ by querying $O_{A_\lambda,\epsilon}$, $O_{D_\lambda,\epsilon}$
  - if true, we are done
  - Else: update H to penalize infeasible directions. Given the separating hyperplane P, update $H \leftarrow h + \dfrac{\epsilon}{16}P$

- $\rho_H \leftarrow \dfrac{e^{-H}}{\mathrm{tr}(e^{-H})}$.

Theorem 2.1 (arXiv: 1909.04613):

Hamiltonian updates find an approximately feasible point in at most:

$$T = \lceil 64\frac{\log(n)}{\epsilon^2}\rceil + 1$$

iterations, otherwise the problem is declared infeasible.

Proof Ideas:
The relative entropy between $\rho_0 = I/n$ and any feasible point $\rho*$ is bounded:

$$S(\rho * ||\rho_0) \leq \log(n) \tag{18.1}$$

We want to show that each iteration makes constant progress in relative entropy: (let $\rho* =$ feasible point).

$$S(\rho * ||\rho_{t+1}) - S(\rho * ||\rho_t) \leq -\frac{\epsilon^2}{64} \tag{18.2}$$

Convergence occurs after at most T steps or $S(\rho * ||\rho_\tau) < 0$, which is impossible by the definition of relative entropy.

Proof Procedures:

Suppose there exists a feasible point ρ^*. Let

$$\rho_t = \frac{\exp(-H_t)}{\mathrm{Tr}(\exp(-H_t))}. \tag{18.3}$$

Distance at time $t = 0$:

$$S(\rho^* \| \rho_0) = \mathrm{Tr}(\rho^*(\log \rho^* - \log \rho_0)) \leq \log(n). \tag{18.4}$$

Improvement at every step:

$$S(\rho^* \| \rho_t) - S(\rho^* \| \rho_{t+1}) = \mathrm{Tr}(\rho^*(\log \rho_t - \log \rho_{t+1})). \tag{18.5}$$

Expanding:

$$\mathrm{Tr}(\rho^*(\log \rho_t - \log \rho_{t+1})) = \mathrm{Tr}\left(\rho^*\left(-H_t - \log \mathrm{Tr}(\exp(-H_t)) + H_{t+1} + \log \mathrm{Tr}(\exp(-H_{t+1}))\right)\right). \tag{18.6}$$

Simplify:

$$= \mathrm{Tr}(\rho^*(H_{t+1} - H_t)) + \log\left(\frac{\mathrm{Tr}(\exp(-H_{t+1}))}{\mathrm{Tr}(\exp(-H_t))}\right). \tag{18.7}$$

Recall update step:

$$H_{t+1} = H_t + \frac{\epsilon}{16} P_t. \tag{18.8}$$

Substituting:

$$= \frac{\epsilon}{16}\,\mathrm{Tr}(\rho^* P_t) - \log\left(\frac{\mathrm{Tr}\left(\exp\left(-H_{t+1} + \frac{\epsilon}{16} P_t\right)\right)}{\mathrm{Tr}(\exp(-H_{t+1}))}\right). \tag{18.9}$$

This term (the second part) is labeled as the "bad boi", which we will work out in detail:

$$\log\left(\frac{\mathrm{Tr}\left(\exp\left(-H_{t+1}+\frac{\epsilon}{16}P_t\right)\right)}{\mathrm{Tr}(\exp(-H_{t+1}))}\right). \tag{18.10}$$

Useful Facts to Analyze "Bad Boi"

1. Peierls-Bogoliubov inequality:

$$\log(\mathrm{Tr}(\exp(F+G))) \geq \mathrm{Tr}(F\exp(G)). \tag{18.11}$$

2. Trace scaling with scalar:

$$\mathrm{Tr}\left(\frac{\exp(-H)}{c}\right) = \mathrm{Tr}\left(\exp(-H)\cdot e^{-\log c}I\right) = \mathrm{Tr}\left(\exp\left(-H-(\log c)I\right)\right). \tag{18.12}$$

Analyzing "Bad Boi"

By fact (2), "bad boi" becomes:

$$\log\left(\mathrm{Tr}\left(\exp\left(-H_{t+1}-\log\left(\mathrm{Tr}(\exp(-H_{t+1}))\right)I+\frac{\epsilon}{16}P_t\right)\right)\right). \tag{18.13}$$

By fact (1):

$$\geq \mathrm{Tr}\left(\frac{\epsilon}{16}P_t\cdot\exp\left(-H_{t+1}-\log\left(\mathrm{Tr}(\exp(-H_{t+1}))\right)I\right)\right). \tag{18.14}$$

Simplify:

$$= \frac{\epsilon}{16}\mathrm{Tr}\left(P_t\cdot\frac{\exp(-H_{t+1})}{\mathrm{Tr}(\exp(-H_{t+1}))}\right) = \frac{\epsilon}{16}\mathrm{Tr}(P_t\rho_{t+1}). \tag{18.15}$$

Continuing from Earlier:

$$\begin{aligned} S(\rho^*\|\rho_{t+1}) - S(\rho^*\|\rho_t) &= \frac{\epsilon}{16}\mathrm{Tr}\left(P_t(\rho^*-\rho_{t+1})\right) && (18.16)\\ &\leq \frac{\epsilon}{16}\mathrm{Tr}(P_t(\rho_t-\rho_{t+1})) - \frac{\epsilon}{16}\mathrm{Tr}(P_t(\rho_t-\rho*)) && (18.17)\\ &\leq \frac{\epsilon}{16}(||P_t||\,||\rho_t-\rho_{t+1}||_{tr}-\frac{\epsilon}{2}) && (18.18) \end{aligned}$$

Using the fact that for Hermitian matrices H_1, H_2:

$$\left\|\frac{\exp(H_1)}{\mathrm{Tr}(\exp(H_1))}-\frac{\exp(H_2)}{\mathrm{Tr}(\exp(H_2))}\right\|_1 \leq 2(\exp\left(\|H_1-H_2\|\right)-1), \tag{18.19}$$

we have:

$$\|\rho_t-\rho_{t+1}\|_1 \leq 2\left(\exp\left(\frac{\epsilon}{16}\|P_t\|\right)-1\right). \tag{18.20}$$

Since $\|P_t\| \leq 1$, this simplifies to:

$$\|\rho_t-\rho_{t+1}\|_1 \leq \frac{\epsilon}{4}. \tag{18.21}$$

Substituting back:

$$S(\rho^*\|\rho_{t+1}) - S(\rho^*\|\rho_t) \leq \frac{\epsilon}{16}\left(\frac{\epsilon}{4}-\frac{\epsilon}{2}\right) = -\frac{\epsilon^2}{64}. \tag{18.22}$$